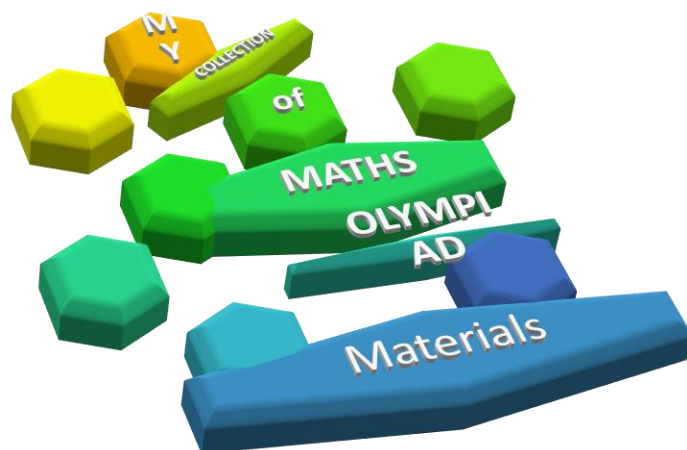


# *My Collection of Maths Olympiad Materials*

— **BENNY VARGHESE**



Hi

Am BENNY VARGHESE PGT MATHEMATICS JNV PATHANAMTHITTA KERALA

My collection of Maths Olympiad Materials is given below :

The syllabus for Mathematical Olympiad (regional, national and international) is pre-degree college mathematics. The areas covered are arithmetic of integers, geometry, quadratic equations and expressions, trigonometry, co-ordinate geometry, system of linear equations, permutations and combination, factorization of polynomial, inequalities, elementary combinatorics, probability theory and number theory, finite series and complex numbers and elementary graph theory. The syllabus does not include calculus and statistics. The major areas from which problems are given are algebra, combinatorics, geometry and number theory . The syllabus is in a sense spread over Class XI to Class XII levels, but the problems under each topic involve high level of difficulty and sophistication. The difficulty level increases from RMO to INMO to IMO.

## Major Change in Mathematical Olympiads Programme 2017-2018

Queries regarding PRMO may be sent by email to [prmo@hbcse.tifr.res.in](mailto:prmo@hbcse.tifr.res.in). Queries will not be replied to individually, but via the FAQ section on this website.

### Important Information concerning the Mathematics Olympiad Programme:

1. The National Board for Higher Mathematics (NBHM) which is in charge of the mathematical olympiad activity, has entrusted its implementation to the Homi Bhabha Centre for Science Education (HBCSE) to be carried out in coordination with the MO-Cell and the National Coordinator appointed by NBHM, supervised by a Committee constituted by NBHM. All major policy decisions with regard to the Mathematical Olympiads programme are made with the concurrence of NBHM. The following MAJOR changes are announced this year.
2. A pre-Regional Mathematical Olympiad (PRMO) exam will be held on August 20, 2017 at different centres all over India. In order to be eligible for the RMO it is mandatory that students qualify through this centrally administered PRMO. No other independently administered examinations will be recognized from this year onwards.
3. Eligibility: All Indian students who are born on or after August 1, 1998 and, in addition, are in Class XI or below are eligible to appear for the PRMO 2017.
4. The PRMO will be a machine-correctable test of 30 questions. Each question has an answer which is a number with one or two digits. Sample PRMO questions and Sample OMR sheet showing marked answers can be downloaded from [here](#).
5. The PRMO exam will be organized this year by IAPT (the Indian Association of Physics Teachers), the same association that also organizes the National Standards Examination, which is the first step for participation in the International Physics, Chemistry, Biology, Astronomy and Junior Science olympiads. The website for PRMO is available at <http://www.iapt.org/in/>

6. A link to a portal is available at the IAPT webpage where schools can register as centres; this portal will be open from May 20 to June 20, 2017. Any school with at least 5 registrations can register on the portal as a “registered centre”. However, the list of approved exam centres will be issued by IAPT. Students who register through a given school which is a registered centre may be re-assigned to another nearby exam centre.
7. All Kendriya Vidyalaya, Jawahar Navodaya Vidyalaya and Atomic Energy Central Schools may register as registered centres regardless of the number of students.
8. The list of registered centres will be available on the IAPT website from June 20, 2017.
9. Students can approach one of the approved registered centres and through the centre register for the PRMO exam between June 20 and July 25, 2017 for a fee of Rs. 200. Kendriya Vidyalaya Sangathan has made its schools available as centres free of cost; for KV students there is a provision to register with payment of a fee of Rs. 100.
10. A carbon copy of the answersheet (OMR) will be given to the student after the exam; students can check their answers as against the answers which will be uploaded on the HBCSE website.
11. The top 300 students from each region will be eligible to write the Regional Mathematics Olympiad exam. For the KV, CBSE, JNV schools which have a countrywide coverage, the number to be selected is upto 5 per cent of the number of registrations or 300.
12. Each regional co-ordinator (including the co-ordinators for the groups KV, CBSE and JNV) will receive the list of students eligible to write RMO by September 15th. Each Regional Co-ordinator shall conduct the RMO in her/his region on October 8<sup>th</sup>, 2017.
13. Based on the RMO exam, the final list of 30 top students plus the next 5 girl students will be sent by each regional co-ordinator to HBCSE by November 30, 2017. This should be done after completing re-evaluation (provision for which is mandatory).
14. HBCSE will announce the list of students eligible to appear for INMO 2018 from all regions by December 07, 2017.
15. The INMO exam will be held on January 21, 2018 and will be conducted by each regional co-ordinator and the papers sent to HBCSE.
16. The regional co-ordinators will be sent brochures/posters and will be requested to advertise the information on their webpages or homepages. As there are major changes, students will be unaware of these and the support of Regional co-ordinators in this regard is crucial.
17. This is a preliminary announcement and more details will be announced and communicated to all regional co-ordinators in the next few days. Regional co-ordinators are requested to circulate the announcement and the weblink widely.

The center registration and student enrollment for all National Standard Examinations will start from August 15th onwards. This year poster and brochure will be available shortly. For details as of now please refer to last years students brochure.

## ANNOUNCEMENT FOR PRE RMO

IAPT in association with National Board for Higher Mathematics (NBHM) is happy to announce the first ever Pre Regional Mathematics Olympiad PRMO as the first selection examination for the team to represent India in International Mathematics Olympiad. For information please visit <http://olympiads.hbcse.tifr.res.in/?p=1447>

### Important dates to note

Registration of Centers 20th MAY -20th JUNE 2017

Registration of Candidates 20th JUNE- 25th JULY 2017

Down loading of Admit Cards by the Center in-charge 1 AUG -12th AUG 2017

Date of Examination 20th AUG 2017 between 10 AM and 12.30 PM

**PLEASE NOTE THE REGISTRATION FOR NATIONAL STANDARD EXAMINATION WILL BE ANNOUNCED LATTER.**

The Indian Association of Physics Teachers (IAPT) was established in the year 1984 by the great visionary, (Late) Dr. D. P. Khandelwal, with active support from some Physics teachers, with the aim of upgrading the quality of Physics teaching and Physics teachers at all levels. It has now grown into a major organisation with about 6500 life members spread all over about 1500 organisations throughout the country including about 100 members from abroad. The members include school, college, university teachers, research workers, science administrators and science savvy enthusiasts. For its grass root working, the country is divided into 20 regions, each with a regional council. The apex executive council, co-ordinates and directs the effort at the national level.

All IAPT work is voluntary, No remuneration is paid to its members for any IAPT activity.

#### **NEW ADDRESS OF THE EXAMINATION OFFICE**

From 1st July 2016,the IAPT examinations will be at the following address

**Prof. G. Venkatesh**

**IAPT Bangalore Centre 2**

**Indian Academy Degree**

**College Hennur Cross, Hennur**

**Main Road Bangalore-560 043**

**e- mail : [iapt.nse@gmail.com](mailto:iapt.nse@gmail.com)**

**Helpline Number. 080 - 49087030**

# National Board for Higher Mathematics & Homi Bhabha Centre for Science Education



## Mathematical Olympiad 2017-18

*Winning Olympiad Medals – a dream of young achievers*

### NOTE

In 2017-18 PRMO has been introduced as the first step of the Mathematical Olympiad Programme



### Pre-Regional Mathematical Olympiad (PRMO)

For information please visit: <http://olympiads.hbcse.tifr.res.in>

For registration please visit: <http://www.iapt.org.in>

#### Important dates for PRMO 2017

Registration of Centers : 20 May - 20 June 2017

Registration of Candidates : 20 June - 25 July 2017

Downloading of Admit Cards by the Center in-charge : 1 August - 12 August 2017

Date of Examination : 20 August 2017

Time of Examination : 10 AM - 12.30 PM

These are the only Olympiads that lead to participation in the International Mathematical Olympiad (IMO). No other contests are recognised for this purpose.

**Homi Bhabha Centre for Science Education**

Tata Institute of Fundamental Research

V. N. Purav Marg, Mankhurd, Mumbai - 400 088

Tel: 022 - 2556 2132, 2556 7711, 2558 0036 Fax: 022 - 2558 5660, 2556 6803

### Sample Questions for PRMO 2017

1. Two positive integers  $a$  and  $b$  are such that  $a + b = \frac{a}{b} + \frac{b}{a}$ . What is the value of  $a^2 + b^2$ ?  
[Ans: 02]
2. The equations  $x^2 - 4x + k = 0$  and  $x^2 + kx - 4 = 0$ , where  $k$  is a real number, have exactly one common root. What is the value of  $k$ ? [Ans: 03]
3. Let  $P(x)$  be a non-zero polynomial with integer coefficients. If  $P(n)$  is divisible by  $n$  for each positive integer  $n$ , what is the value of  $P(0)$ ? [Ans: 00]
4. A natural number  $k$  is such that  $k^2 < 2014 < (k + 1)^2$ . What is the largest prime factor of  $k$ ? [Ans: 11]
5. How many two-digit positive integers  $N$  have the property that the sum of  $N$  and the number obtained reversing the order of the digits of  $N$  is a perfect square? [Ans: 08]
6. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12? [Ans: 84]
7. In rectangle  $ABCD$ ,  $AB = 8$  and  $BC = 20$ . Let  $P$  be a point on  $AD$  such that  $\angle BPC = 90^\circ$ . If  $r_1, r_2, r_3$  are the radii of the incircles of triangles  $APB$ ,  $BPC$  and  $CPD$ , what is the value of  $r_1 + r_2 + r_3$ ? [Ans: 08]
8. Let  $n$  be the largest integer that is the product of exactly 3 distinct prime numbers,  $x, y$  and  $10x + y$ , where  $x$  and  $y$  are digits. What is the sum of the digits of  $n$ ? [Ans: 12]
9. A subset  $B$  of the set of first 100 positive integers has the property that no two elements of  $B$  sum to 125. What is the maximum possible number of elements in  $B$ ? [Ans: 62]
10. The circle  $\omega$  touches the circle  $\Omega$  internally at  $P$ . The centre  $O$  of  $\Omega$  is outside  $\omega$ . Let  $XY$  be a diameter of  $\Omega$  which is also tangent to  $\omega$ . Assume  $PY > PX$ . Let  $PY$  intersect  $\omega$  at  $Z$ . If  $YZ = 2PZ$ , what is the magnitude of  $\angle PYX$  in degrees? [Ans: 15]

# OMR ANSWER SHEET

**ORIGINAL COPY**

### 1. STUDENT'S NAME

**STUDENT NAME**

2. ROLL NO.

A	P	K	0	1	2	3	4	4	4
(A)	(A)	(●)	(●)	(0)	(0)	(0)	(0)	(0)	(0)
(B)	(B)	(M)	(1)	(●)	(1)	(1)	(1)	(1)	(1)
(C)	(C)		(2)	(2)	(●)	(2)	(2)	(1)	(2)
(D)	(D)		(3)	(3)	(3)	(●)	(3)	(3)	(3)
(E)	(E)		(4)	(4)	(4)	(4)	(●)	(●)	(●)
(F)	(F)		(5)	(5)	(5)	(5)	(5)	(5)	(5)
(G)	(G)		(6)	(6)	(6)	(6)	(6)	(6)	(6)
(H)	(H)		(7)	(7)	(7)	(7)	(7)	(7)	(7)
(I)	(I)		(8)	(8)	(8)	(8)	(8)	(8)	(8)
(J)	(J)		(9)	(9)	(9)	(9)	(9)	(9)	(9)
(K)	(K)								
(L)	(L)								
(M)	(M)								
(N)	(N)								
(O)	(O)								
(P)	(●)								
(Q)	(Q)								
(R)	(R)								
(S)	(S)								
(T)	(T)								
(U)	(U)								
(V)	(V)								
(W)	(W)								
(X)	(X)								
(Y)	(Y)								
(Z)	(Z)								

### 3. DATE OF BIRTH

08      11      2004

0 1 2 3 4 5 6 7 8 9

0 1 2 3 4 5 6 7 8 9

0 1 2 3 4 5 6 7 8 9

1998  
1999  
2000  
2001  
2002  
2003  
2004

#### 4. STANDARD

IX

VIII VIII

IX

X X

XI XI

## INSTRUCTIONS FOR MARKING

1. MARKING SHOULD BE DARK AND COMPLETELY FILL USING BLUE/BLACK BALL POINT PEN ONLY AS SHOWN BELOW.

## WRONG METHODS

CORRECT METHOD

0 1 2 3 4 5 6 7 8 9

2. MARK YOUR ANSWER ONLY IN THE APPROPRIATE SPACE.  
3. ANY STRAY MARKING/TAMPERING WITH THE ANSWER SHEET WOULD BE TREATED AS MALPRACTICE.

## 5. GENDER

SHEET NO.



## 6. NAME OF THE INSTITUTE

**KENDRIYA VIDYALAYA NO. 1**

## MARK YOUR ANSWERS

Q. NO	ANSWERS
1	<div>● ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>○ ① ② ● ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div>
2	<div>● ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>○ ① ② ③ ● ④ ⑤ ⑥ ⑦ ⑧ ⑨</div>
3	<div>● ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>● ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div>
4	<div>○ ① ● ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>○ ① ● ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div>
5	<div>● ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>○ ① ② ③ ④ ⑤ ⑥ ⑦ ● ⑧ ⑨</div>
6	<div>○ ① ② ③ ④ ⑤ ⑥ ⑦ ● ⑧ ⑨</div> <div>○ ① ② ③ ● ④ ⑤ ⑥ ⑦ ⑧ ⑨</div>
7	<div>● ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>○ ① ② ③ ④ ⑤ ⑥ ⑦ ● ⑧ ⑨</div>
8	<div>○ ① ● ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>○ ① ● ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div>
9	<div>○ ① ② ③ ④ ⑤ ● ⑦ ⑧ ⑨</div> <div>○ ① ● ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div>
10	<div>○ ① ● ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨</div> <div>○ ① ② ③ ④ ● ⑥ ⑦ ⑧ ⑨</div>

Q. NO	ANSWERS
11	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
12	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
13	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
14	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
15	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
16	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
17	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
18	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
19	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9
20	0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9

Q. NO	ANSWERS
21	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
22	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
23	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
24	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
25	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
26	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
27	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
28	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
29	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>
30	<div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div> <div>Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ Ⓗ Ⓘ Ⓢ</div>

STUDENT'S  
SIGNATURE

INVIGILATOR'S  
SIGNATURE

# NATIONAL BOARD FOR HIGHER MATHEMATICS

A

AND  
HOMI BHABHA CENTRE FOR SCIENCE EDUCATION  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2012

Mumbai Region

October 7, 2012

## QUESTION PAPER SET: A

- | There are 20 questions in this question paper. Each question carries 5 marks.
- | Answer all questions.
- | Time allotted: 2 hours.

## QUESTIONS

1. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly?
2. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?
3. For how many pairs of positive integers  $(x, y)$  is  $x + 3y = 100$ ?
4. The letters  $R$ ,  $M$ , and  $O$  represent whole numbers. If  $R \times M \times O = 240$ ,  $R \times O + M = 46$  and  $R + M \times O = 64$ , what is the value of  $R + M + O$ ?
5. Let  $S_n = n^2 + 20n + 12$ ,  $n$  a positive integer. What is the sum of all possible values of  $n$  for which  $S_n$  is a perfect square?
6. A postman has to deliver five letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how many different ways could he do this so that exactly two of the five houses receive the correct letters?
7. In  $\triangle ABC$ , we have  $AC = BC = 7$  and  $AB = 2$ . Suppose that  $D$  is a point on line  $AB$  such that  $B$  lies between  $A$  and  $D$  and  $CD = 8$ . What is the length of the segment  $BD$ ?
8. In rectangle  $ABCD$ ,  $AB = 5$  and  $BC = 3$ . Points  $F$  and  $G$  are on line segment  $CD$  so that  $DF = 1$  and  $GC = 2$ . Lines  $AF$  and  $BG$  intersect at  $E$ . What is the area of  $\triangle AEB$ ?
9. Suppose that  $4^{X_1} = 5, 5^{X_2} = 6, 6^{X_3} = 7, \dots, 126^{X_{123}} = 127, 127^{X_{124}} = 128$ . What is the value of the product  $X_1 X_2 \dots X_{124}$ ?
10.  $ABCD$  is a square and  $AB = 1$ . Equilateral triangles  $AYB$  and  $CXD$  are drawn such that  $X$  and  $Y$  are inside the square. What is the length of  $XY$ ?
11. Let  $P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$ . What is the largest integer that is a divisor of  $P(n)$  for all positive even integers  $n$ ?
12. If  $\frac{1}{\sqrt{2011 + \sqrt{2011^2 - 1}}} = \sqrt{m} - \sqrt{n}$ , where  $m$  and  $n$  are positive integers, what is the value of  $m + n$ ?

13. If  $a = b - c$ ,  $b = c - d$ ,  $c = d - a$  and  $abcd \neq 0$  then what is the value of  $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$ ?
14.  $O$  and  $I$  are the circumcentre and incentre of  $\triangle ABC$  respectively. Suppose  $O$  lies in the interior of  $\triangle ABC$  and  $I$  lies on the circle passing through  $B$ ,  $O$ , and  $C$ . What is the magnitude of  $\angle BAC$  in degrees?
15. How many non-negative integral values of  $x$  satisfy the equation  $\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]$ ?  
(Here  $[x]$  denotes the greatest integer less than or equal to  $x$ . For example  $[3.4] = 3$  and  $[-2.3] = -3$ .)
16. Let  $N$  be the set of natural numbers. Suppose  $f : N \rightarrow N$  is a function satisfying the following conditions.
- (a)  $f(mn) = f(m)f(n)$ ;
  - (b)  $f(m) < f(n)$  if  $m < n$ ;
  - (c)  $f(2) = 2$ .

What is the value of  $\sum_{k=1}^{20} f(k)$ ?

17. Let  $x_1, x_2, x_3$  be the roots of the equation  $x^3 + 3x + 5 = 0$ . What is the value of the expression

$$\left(x_1 + \frac{1}{x_1}\right) \left(x_2 + \frac{1}{x_2}\right) \left(x_3 + \frac{1}{x_3}\right)?$$

18. What is the sum of the squares of the roots of the equation  $x^2 - 7[x] + 5 = 0$ ?  
(Here  $[x]$  denotes the greatest integer less than or equal to  $x$ . For example  $[3.4] = 3$  and  $[-2.3] = -3$ .)
19. How many integer pairs  $(x, y)$  satisfy  $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$ ?
20.  $PS$  is a line segment of length 4 and  $O$  is the midpoint of  $PS$ . A semicircular arc is drawn with  $PS$  as diameter. Let  $X$  be the midpoint of this arc.  $Q$  and  $R$  are points on the arc  $PXS$  such that  $QR$  is parallel to  $PS$  and the semicircular arc drawn with  $QR$  as diameter is tangent to  $PS$ . What is the area of the region  $QXROQ$  bounded by the two semicircular arcs?

END OF QUESTION PAPER

A

NATIONAL BOARD FOR HIGHER MATHEMATICS  
AND  
HOMI BHABHA CENTRE FOR SCIENCE EDUCATION  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2013  
Mumbai Region

October 20, 2013

QUESTION PAPER SET: A

- 
- There are 20 questions in this question paper. Each question carries 5 marks.
  - Answer all questions.
  - Time allotted: 2 hours.
- 

QUESTIONS

1. What is the smallest positive integer  $k$  such that  $k(3^3 + 4^3 + 5^3) = a^n$  for some positive integers  $a$  and  $n$ , with  $n > 1$ ?
2. Let  $S_n = \sum_{k=0}^n \frac{1}{\sqrt{k+1} + \sqrt{k}}$ . What is the value of  $\sum_{n=1}^{99} \frac{1}{S_n + S_{n-1}}$ ?
3. It is given that the equation  $x^2 + ax + 20 = 0$  has integer roots. What is the sum of all possible values of  $a$ ?
4. Three points  $X, Y, Z$  are on a straight line such that  $XY = 10$  and  $XZ = 3$ . What is the product of all possible values of  $YZ$ ?
5. There are  $n - 1$  red balls,  $n$  green balls and  $n + 1$  blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of  $n$ ?
6. Let  $S(M)$  denote the sum of the digits of a positive integer  $M$  written in base 10. Let  $N$  be the smallest positive integer such that  $S(N) = 2013$ . What is the value of  $S(5N + 2013)$ ?
7. Let Akbar and Birbal together have  $n$  marbles, where  $n > 0$ .  
Akbar says to Birbal, "If I give you some marbles then you will have twice as many marbles as I will have." Birbal says to Akbar, "If I give you some marbles then you will have thrice as many marbles as I will have."  
What is the minimum possible value of  $n$  for which the above statements are true?
8. Let  $AD$  and  $BC$  be the parallel sides of a trapezium  $ABCD$ . Let  $P$  and  $Q$  be the midpoints of the diagonals  $AC$  and  $BD$ . If  $AD = 16$  and  $BC = 20$ , what is the length of  $PQ$ ?

9. In a triangle  $ABC$ , let  $H$ ,  $I$  and  $O$  be the orthocentre, incentre and circumcentre, respectively. If the points  $B$ ,  $H$ ,  $I$ ,  $C$  lie on a circle, what is the magnitude of  $\angle BOC$  in degrees?
10. Carol was given three numbers and was asked to add the largest of the three to the product of the other two. Instead, she multiplied the largest with the sum of the other two, but still got the right answer. What is the sum of the three numbers?
11. Three real numbers  $x$ ,  $y$ ,  $z$  are such that  $x^2 + 6y = -17$ ,  $y^2 + 4z = 1$  and  $z^2 + 2x = 2$ . What is the value of  $x^2 + y^2 + z^2$ ?
12. Let  $ABC$  be an equilateral triangle. Let  $P$  and  $S$  be points on  $AB$  and  $AC$ , respectively, and let  $Q$  and  $R$  be points on  $BC$  such that  $PQRS$  is a rectangle. If  $PQ = \sqrt{3}PS$  and the area of  $PQRS$  is  $28\sqrt{3}$ , what is the length of  $PC$ ?
13. To each element of the set  $S = \{1, 2, \dots, 1000\}$  a colour is assigned. Suppose that for any two elements  $a, b$  of  $S$ , if 15 divides  $a + b$  then they are both assigned the same colour. What is the maximum possible number of distinct colours used?
14. Let  $m$  be the smallest odd positive integer for which  $1 + 2 + \dots + m$  is a square of an integer and let  $n$  be the smallest even positive integer for which  $1 + 2 + \dots + n$  is a square of an integer. What is the value of  $m + n$ ?
15. Let  $A_1, B_1, C_1, D_1$  be the midpoints of the sides of a convex quadrilateral  $ABCD$  and let  $A_2, B_2, C_2, D_2$  be the midpoints of the sides of the quadrilateral  $A_1B_1C_1D_1$ . If  $A_2B_2C_2D_2$  is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of  $ABCD$ ?
16. Let  $f(x) = x^3 - 3x + b$  and  $g(x) = x^2 + bx - 3$ , where  $b$  is a real number. What is the sum of all possible values of  $b$  for which the equations  $f(x) = 0$  and  $g(x) = 0$  have a common root?
17. Let  $S$  be a circle with centre  $O$ . A chord  $AB$ , not a diameter, divides  $S$  into two regions  $R_1$  and  $R_2$  such that  $O$  belongs to  $R_2$ . Let  $S_1$  be a circle with centre in  $R_1$ , touching  $AB$  at  $X$  and  $S$  internally. Let  $S_2$  be a circle with centre in  $R_2$ , touching  $AB$  at  $Y$ , the circle  $S$  internally and passing through the centre of  $S$ . The point  $X$  lies on the diameter passing through the centre of  $S_2$  and  $\angle YXO = 30^\circ$ . If the radius of  $S_2$  is 100 then what is the radius of  $S_1$ ?
18. What is the maximum possible value of  $k$  for which 2013 can be written as a sum of  $k$  consecutive positive integers?
19. In a triangle  $ABC$  with  $\angle BCA = 90^\circ$ , the perpendicular bisector of  $AB$  intersects segments  $AB$  and  $AC$  at  $X$  and  $Y$ , respectively. If the ratio of the area of quadrilateral  $BXYC$  to the area of triangle  $ABC$  is  $13 : 18$  and  $BC = 12$  then what is the length of  $AC$ ?
20. What is the sum (in base 10) of all the natural numbers less than 64 which have exactly three ones in their base 2 representation?

NATIONAL BOARD FOR HIGHER MATHEMATICS  
AND  
HOMI BHABHA CENTRE FOR SCIENCE EDUCATION  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2014  
Mumbai Region

October 12, 2014

QUESTION PAPER SET: A

- 
- There are 20 questions in this question paper. Each question carries 5 marks.
  - Answer all questions.
  - Time allotted: 2.5 hours.
- 

QUESTIONS

1. A natural number  $k$  is such that  $k^2 < 2014 < (k+1)^2$ . What is the largest prime factor of  $k$ ?
2. The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014<sup>th</sup> term of the sequence?
3. Let  $ABCD$  be a convex quadrilateral with perpendicular diagonals. If  $AB = 20$ ,  $BC = 70$  and  $CD = 90$ , then what is the value of  $DA$ ?
4. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle?
5. If real numbers  $a, b, c, d, e$  satisfy

$$a + 1 = b + 2 = c + 3 = d + 4 = e + 5 = a + b + c + d + e + 3,$$

what is the value of  $a^2 + b^2 + c^2 + d^2 + e^2$ ?

6. What is the smallest possible natural number  $n$  for which the equation  $x^2 - nx + 2014 = 0$  has integer roots?
7. If  $x^{(x^4)} = 4$ , what is the value of  $x^{(x^2)} + x^{(x^8)}$ ?
8. Let  $S$  be a set of real numbers with mean  $M$ . If the means of the sets  $S \cup \{15\}$  and  $S \cup \{15, 1\}$  are  $M + 2$  and  $M + 1$ , respectively, then how many elements does  $S$  have?
9. Natural numbers  $k, l, p$  and  $q$  are such that if  $a$  and  $b$  are roots of  $x^2 - kx + l = 0$  then  $a + \frac{1}{b}$  and  $b + \frac{1}{a}$  are the roots of  $x^2 - px + q = 0$ . What is the sum of all possible values of  $q$ ?
10. In a triangle  $ABC$ ,  $X$  and  $Y$  are points on the segments  $AB$  and  $AC$ , respectively, such that  $AX : XB = 1 : 2$  and  $AY : YC = 2 : 1$ . If the area of triangle  $AXY$  is 10 then what is the area of triangle  $ABC$ ?
11. For natural numbers  $x$  and  $y$ , let  $(x, y)$  denote the greatest common divisor of  $x$  and  $y$ . How many pairs of natural numbers  $x$  and  $y$  with  $x \leq y$  satisfy the equation  $xy = x + y + (x, y)$ ?

12. Let  $ABCD$  be a convex quadrilateral with  $\angle DAB = \angle BDC = 90^\circ$ . Let the incircles of triangles  $ABD$  and  $BCD$  touch  $BD$  at  $P$  and  $Q$ , respectively, with  $P$  lying in between  $B$  and  $Q$ . If  $AD = 999$  and  $PQ = 200$  then what is the sum of the radii of the incircles of triangles  $ABD$  and  $BDC$ ?
13. For how many natural numbers  $n$  between 1 and 2014 (both inclusive) is  $\frac{8n}{9999 - n}$  an integer?
14. One morning, each member of Manjul's family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Manjul drank  $1/7$ -th of the total amount of milk and  $2/17$ -th of the total amount of coffee. How many people are there in Manjul's family?
15. Let  $XOY$  be a triangle with  $\angle XOY = 90^\circ$ . Let  $M$  and  $N$  be the midpoints of legs  $OX$  and  $OY$ , respectively. Suppose that  $XN = 19$  and  $YM = 22$ . What is  $XY$ ?
16. In a triangle  $ABC$ , let  $I$  denote the incenter. Let the lines  $AI, BI$  and  $CI$  intersect the incircle at  $P, Q$  and  $R$ , respectively. If  $\angle BAC = 40^\circ$ , what is the value of  $\angle QPR$  in degrees?
17. For a natural number  $b$ , let  $N(b)$  denote the number of natural numbers  $a$  for which the equation  $x^2 + ax + b = 0$  has integer roots. What is the smallest value of  $b$  for which  $N(b) = 20$ ?
18. Let  $f$  be a one-to-one function from the set of natural numbers to itself such that  $f(mn) = f(m)f(n)$  for all natural numbers  $m$  and  $n$ . What is the least possible value of  $f(999)$ ?
19. Let  $x_1, x_2, \dots, x_{2014}$  be real numbers different from 1, such that  $x_1 + x_2 + \dots + x_{2014} = 1$  and

$$\frac{x_1}{1 - x_1} + \frac{x_2}{1 - x_2} + \dots + \frac{x_{2014}}{1 - x_{2014}} = 1.$$

What is the value of

$$\frac{x_1^2}{1 - x_1} + \frac{x_2^2}{1 - x_2} + \frac{x_3^2}{1 - x_3} + \dots + \frac{x_{2014}^2}{1 - x_{2014}} ?$$

20. What is the number of ordered pairs  $(A, B)$  where  $A$  and  $B$  are subsets of  $\{1, 2, \dots, 5\}$  such that neither  $A \subseteq B$  nor  $B \subseteq A$ ?

Pre-RMO 2014 results will be put up on HBCSE website by 31/10/2014.

**NATIONAL BOARD FOR HIGHER MATHEMATICS**  
AND  
**HOMI BHABHA CENTRE FOR SCIENCE EDUCATION**  
**TATA INSTITUTE OF FUNDAMENTAL RESEARCH**

**Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2015**  
**Mumbai Region**

**October 4, 2015**

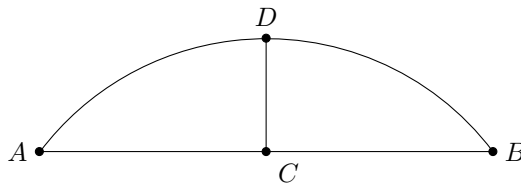
**QUESTION PAPER SET: A**

---

- There are 20 questions in this question paper. Each question carries 5 marks.
  - Answer all questions.
  - Time allotted: 2.5 hours.
- 

**QUESTIONS**

1. A man walks a certain distance and rides back in  $3\frac{3}{4}$  hours; he could ride both ways in  $2\frac{1}{2}$  hours. How many hours would it take him to walk both ways? [5]
2. Positive integers  $a$  and  $b$  are such that  $a + b = a/b + b/a$ . What is the value of  $a^2 + b^2$ ? [2]
3. The equations  $x^2 - 4x + k = 0$  and  $x^2 + kx - 4 = 0$ , where  $k$  is a real number, have exactly one common root. What is the value of  $k$ ? [3]
4. Let  $P(x)$  be a non-zero polynomial with integer coefficients. If  $P(n)$  is divisible by  $n$  for each positive integer  $n$ , what is the value of  $P(0)$ ? [0]
5. How many line segments have both their endpoints located at the vertices of a given cube? [28]
6. Let  $E(n)$  denote the sum of the even digits of  $n$ . For example,  $E(1243) = 2 + 4 = 6$ . What is the value of  $E(1) + E(2) + E(3) + \cdots + E(100)$ ? [400]
7. How many two-digit positive integers  $N$  have the property that the sum of  $N$  and the number obtained by reversing the order of the digits of  $N$  is a perfect square? [8]
8. The figure below shows a broken piece of a circular plate made of glass.



$C$  is the midpoint of  $AB$ , and  $D$  is the midpoint of arc  $AB$ . Given that  $AB = 24$  cm and  $CD = 6$  cm, what is the radius of the plate in centimetres? (The figure is not drawn to scale.) [15]

9. A  $2 \times 3$  rectangle and a  $3 \times 4$  rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? [25]
10. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12 ? [84]
11. In rectangle  $ABCD$ ,  $AB = 8$  and  $BC = 20$ . Let  $P$  be a point on  $AD$  such that  $\angle BPC = 90^\circ$ . If  $r_1, r_2, r_3$  are the radii of the incircles of triangles  $APB$ ,  $BPC$  and  $CPD$ , what is the value of  $r_1 + r_2 + r_3$ ? [8]
12. Let  $a, b$ , and  $c$  be real numbers such that  $a - 7b + 8c = 4$  and  $8a + 4b - c = 7$ . What is the value of  $a^2 - b^2 + c^2$ ? [1]
13. Let  $n$  be the largest integer that is the product of exactly 3 distinct prime numbers,  $x, y$  and  $10x + y$ , where  $x$  and  $y$  are digits. What is the sum of the digits of  $n$ ? [12]
14. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party? [12]
15. If  $3^x + 2^y = 985$  and  $3^x - 2^y = 473$ , what is the value of  $xy$ ? [48]
16. In acute-angled triangle  $ABC$ , let  $D$  be the foot of the altitude from  $A$ , and  $E$  be the midpoint of  $BC$ . Let  $F$  be the midpoint of  $AC$ . Suppose  $\angle BAE = 40^\circ$ . If  $\angle DAE = \angle DFE$ , what is the magnitude of  $\angle ADF$  in degrees? [40]
17. A subset  $B$  of the set of first 100 positive integers has the property that no two elements of  $B$  sum to 125. What is the maximum possible number of elements in  $B$ ? [62]
18. Let  $a, b$  and  $c$  be such that  $a + b + c = 0$  and

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$$

is defined. What is the value of  $P$ ? [1]

19. The circle  $\omega$  touches the circle  $\Omega$  internally at  $P$ . The centre  $O$  of  $\Omega$  is outside  $\omega$ . Let  $XY$  be a diameter of  $\Omega$  which is also tangent to  $\omega$ . Assume  $PY > PX$ . Let  $PY$  intersect  $\omega$  at  $Z$ . If  $YZ = 2PZ$ , what is the magnitude of  $\angle PYX$  in degrees? [15]
20. The digits of a positive integer  $n$  are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when  $n$  is divided by 37? [217]

**NATIONAL BOARD FOR HIGHER MATHEMATICS**  
AND  
**HOMI BHABHA CENTRE FOR SCIENCE EDUCATION**  
**TATA INSTITUTE OF FUNDAMENTAL RESEARCH**

**Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2015**  
**Mumbai Region**

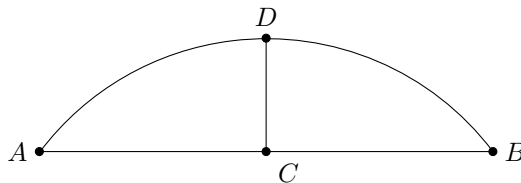
**October 4, 2015**

**QUESTION PAPER SET: B**

- 
- There are 20 questions in this question paper. Each question carries 5 marks.
  - Answer all questions.
  - Time allotted: 2.5 hours.
- 

**QUESTIONS**

1. A man walks a certain distance and rides back in  $3\frac{3}{4}$  hours; he could ride both ways in  $2\frac{1}{2}$  hours. How many hours would it take him to walk both ways? [5]
2. The equations  $x^2 - 4x + k = 0$  and  $x^2 + kx - 4 = 0$ , where  $k$  is a real number, have exactly one common root. What is the value of  $k$ ? [3]
3. Positive integers  $a$  and  $b$  are such that  $a + b = a/b + b/a$ . What is the value of  $a^2 + b^2$ ? [2]
4. How many line segments have both their endpoints located at the vertices of a given cube? [28]
5. Let  $P(x)$  be a non-zero polynomial with integer coefficients. If  $P(n)$  is divisible by  $n$  for each positive integer  $n$ , what is the value of  $P(0)$ ? [0]
6. How many two-digit positive integers  $N$  have the property that the sum of  $N$  and the number obtained by reversing the order of the digits of  $N$  is a perfect square? [8]
7. Let  $E(n)$  denote the sum of the even digits of  $n$ . For example,  $E(1243) = 2 + 4 = 6$ . What is the value of  $E(1) + E(2) + E(3) + \cdots + E(100)$ ? [400]
8. The figure below shows a broken piece of a circular plate made of glass.



$C$  is the midpoint of  $AB$ , and  $D$  is the midpoint of arc  $AB$ . Given that  $AB = 24$  cm and  $CD = 6$  cm, what is the radius of the plate in centimetres? (The figure is not drawn to scale.) [15]

9. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12 ? [84]
10. A  $2 \times 3$  rectangle and a  $3 \times 4$  rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? [25]
11. Let  $a$ ,  $b$ , and  $c$  be real numbers such that  $a - 7b + 8c = 4$  and  $8a + 4b - c = 7$ . What is the value of  $a^2 - b^2 + c^2$ ? [1]
12. In rectangle  $ABCD$ ,  $AB = 8$  and  $BC = 20$ . Let  $P$  be a point on  $AD$  such that  $\angle BPC = 90^\circ$ . If  $r_1$ ,  $r_2$ ,  $r_3$  are the radii of the incircles of triangles  $APB$ ,  $BPC$  and  $CPD$ , what is the value of  $r_1 + r_2 + r_3$ ? [8]
13. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party? [12]
14. If  $3^x + 2^y = 985$  and  $3^x - 2^y = 473$ , what is the value of  $xy$ ? [48]
15. Let  $n$  be the largest integer that is the product of exactly 3 distinct prime numbers,  $x$ ,  $y$  and  $10x + y$ , where  $x$  and  $y$  are digits. What is the sum of the digits of  $n$ ? [12]
16. In acute-angled triangle  $ABC$ , let  $D$  be the foot of the altitude from  $A$ , and  $E$  be the midpoint of  $BC$ . Let  $F$  be the midpoint of  $AC$ . Suppose  $\angle BAE = 40^\circ$ . If  $\angle DAE = \angle DFE$ , what is the magnitude of  $\angle ADF$  in degrees? [40]
17. Let  $a$ ,  $b$  and  $c$  be such that  $a + b + c = 0$  and

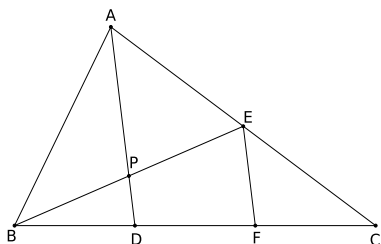
$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$$

is defined. What is the value of  $P$ ? [1]

18. A subset  $B$  of the set of first 100 positive integers has the property that no two elements of  $B$  sum to 125. What is the maximum possible number of elements in  $B$ ? [62]
19. The digits of a positive integer  $n$  are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when  $n$  is divided by 37? [217]
20. The circle  $\omega$  touches the circle  $\Omega$  internally at  $P$ . The centre  $O$  of  $\Omega$  is outside  $\omega$ . Let  $XY$  be a diameter of  $\Omega$  which is also tangent to  $\omega$ . Assume  $PY > PX$ . Let  $PY$  intersect  $\omega$  at  $Z$ . If  $YZ = 2PZ$ , what is the magnitude of  $\angle PYX$  in degrees? [15]

## Problems and Solutions: CRMO-2012, Paper 1

1. Let  $ABC$  be a triangle and  $D$  be a point on the segment  $BC$  such that  $DC = 2BD$ . Let  $E$  be the mid-point of  $AC$ . Let  $AD$  and  $BE$  intersect in  $P$ . Determine the ratios  $BP/PE$  and  $AP/PD$ .



**Solution:** Let  $F$  be the midpoint of  $DC$ , so that  $D, F$  are points of trisection of  $BC$ . Now in triangle  $CAD$ ,  $F$  is the mid-point of  $CD$  and  $E$  is that of  $CA$ . Hence  $CF/FD = 1 = CE/EA$ . Thus  $EF \parallel AD$ . Hence we find that  $EF \parallel PD$ . Hence  $BP/PE = BD/DF$ . But  $BD = DF$ . We obtain  $BP/PE = 1$ .

In triangle  $ACD$ , since  $EF \parallel AD$  we get  $EF/AD = CF/CD = 1/2$ . Thus  $AD = 2EF$ . But  $PD/EF = BD/BF = 1/2$ . Hence  $EF = 2PD$ . Therefore This gives

$$AP = AD - PD = 3PD.$$

We obtain  $AP/PD = 3$ .

(Coordinate geometry proof is also possible.)

2. Let  $a, b, c$  be positive integers such that  $a$  divides  $b^3$ ,  $b$  divides  $c^3$  and  $c$  divides  $a^3$ . Prove that  $abc$  divides  $(a + b + c)^{13}$ .

**Solution:** If a prime  $p$  divides  $a$ , then  $p \mid b^3$  and hence  $p \mid b$ . This implies that  $p \mid c^3$  and hence  $p \mid c$ . Thus every prime dividing  $a$  also divides  $b$  and  $c$ . By symmetry, this is true for  $b$  and  $c$  as well. We conclude that  $a, b, c$  have the same set of prime divisors.

Let  $p^x \parallel a$ ,  $p^y \parallel b$  and  $p^z \parallel c$ . (Here we write  $p^x \parallel a$  to mean  $p^x \mid a$  and  $p^{x+1} \nmid a$ .) We may assume  $\min\{x, y, z\} = x$ . Now  $b \mid c^3$  implies that  $y \leq 3z$ ;  $c \mid a^3$  implies that  $z \leq 3x$ . We obtain

$$y \leq 3z \leq 9x.$$

Thus  $x + y + z \leq x + 3x + 9x = 13x$ . Hence the maximum power of  $p$  that divides  $abc$  is  $x + y + z \leq 13x$ . Since  $x$  is the minimum among  $x, y, z$ , whence  $p^x$  divides each of  $a, b, c$ . Hence  $p^x$  divides  $a + b + c$ . This implies that  $p^{13x}$  divides  $(a + b + c)^{13}$ . Since  $x + y + z \leq 13x$ , it follows that  $p^{x+y+z}$  divides  $(a + b + c)^{13}$ . This is true of any prime  $p$  dividing  $a, b, c$ . Hence  $abc$  divides  $(a + b + c)^{13}$ .

3. Let  $a$  and  $b$  be positive real numbers such that  $a + b = 1$ . Prove that

$$a^a b^b + a^b b^a \leq 1.$$

**Solution:** Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$

Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$

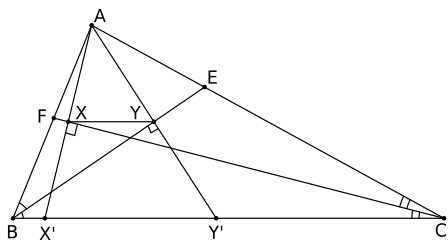
Now if  $a \leq b$ , then  $a^a \leq b^a$  and  $a^b \leq b^b$ . If  $a \geq b$ , then  $a^a \geq b^a$  and  $a^b \geq b^b$ . Hence the product is nonnegative for all positive  $a$  and  $b$ . It follows that

$$a^a b^b + a^b b^a \leq 1.$$

4. Let  $X = \{1, 2, 3, \dots, 10\}$ . Find the the number of pairs  $\{A, B\}$  such that  $A \subseteq X$ ,  $B \subseteq X$ ,  $A \neq B$  and  $A \cap B = \{2, 3, 5, 7\}$ .

**Solution:** Let  $A \cup B = Y$ ,  $B \setminus A = M$ ,  $A \setminus B = N$  and  $X \setminus Y = L$ . Then  $X$  is the disjoint union of  $M$ ,  $N$ ,  $L$  and  $A \cap B$ . Now  $A \cap B = \{2, 3, 5, 7\}$  is fixed. The remaining six elements  $1, 4, 6, 8, 9, 10$  can be distributed in any of the remaining sets  $M$ ,  $N$ ,  $L$ . This can be done in  $3^6$  ways. Of these if all the elements are in the set  $L$ , then  $A = B = \{2, 3, 5, 7\}$  and which this case has to be deleted. Hence the total number of pairs  $\{A, B\}$  such that  $A \subseteq X$ ,  $B \subseteq X$ ,  $A \neq B$  and  $A \cap B = \{2, 3, 5, 7\}$  is  $3^6 - 1$ .

5. Let  $ABC$  be a triangle. Let  $BE$  and  $CF$  be internal angle bisectors of  $\angle B$  and  $\angle C$  respectively with  $E$  on  $AC$  and  $F$  on  $AB$ . Suppose  $X$  is a point on the segment  $CF$  such that  $AX \perp CF$ ; and  $Y$  is a point on the segment  $BE$  such that  $AY \perp BE$ . Prove that  $XY = (b + c - a)/2$  where  $BC = a$ ,  $CA = b$  and  $AB = c$ .



**Solution:** Produce  $AX$  and  $AY$  to meet  $BC$  at  $X'$  and  $Y'$  respectively. Since  $BE$  bisects  $\angle ABY'$  and  $BE \perp AY'$  it follows that  $BA = BY'$  and  $AY = YY'$ . Similarly,  $CA = CX'$  and  $AX = XX'$ . Thus  $X$  and  $Y$  are mid-points of  $AX'$  and  $AY'$  respectively. By mid-point theorem  $XY = X'Y'/2$ . But

$$X'Y' = X'C + Y'B - BC = AC + AB - BC = b + c - a.$$

Hence  $XY = (b + c - a)/2$ .

6. Let  $a$  and  $b$  be real numbers such that  $a \neq 0$ . Prove that not all the roots of  $ax^4 + bx^3 + x^2 + x + 1 = 0$  can be real.

**Solution:** Let  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  be the roots of  $ax^4 + bx^3 + x^2 + x + 1 = 0$ . Observe none of these is zero since their product is  $1/a$ . Then the roots of  $x^4 + x^3 + x^2 + bx + a = 0$  are

$$\beta_1 = \frac{1}{\alpha_1}, \beta_2 = \frac{1}{\alpha_2}, \beta_3 = \frac{1}{\alpha_3}, \beta_4 = \frac{1}{\alpha_4}.$$

We have

$$\sum_{j=1}^4 \beta_j = -1, \quad \sum_{1 \leq j < k \leq 4} \beta_j \beta_k = 1.$$

Hence

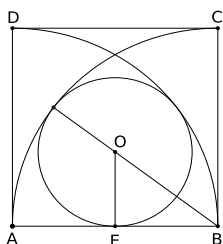
$$\sum_{j=1}^4 \beta_j^2 = \left( \sum_{j=1}^4 \beta_j \right)^2 - 2 \left( \sum_{1 \leq j < k \leq 4} \beta_j \beta_k \right) = 1 - 2 = -1.$$

This shows that not all  $\beta_j$  can be real. Hence not all  $\alpha_j$ 's can be real.

————00000————

## Problems and Solutions: CRMO-2012, Paper 2

1. Let  $ABCD$  be a unit square. Draw a quadrant of a circle with  $A$  as centre and  $B, D$  as end points of the arc. Similarly, draw a quadrant of a circle with  $B$  as centre and  $A, C$  as end points of the arc. Inscribe a circle  $\Gamma$  touching the arc  $AC$  internally, the arc  $BD$  internally and also touching the side  $AB$ . Find the radius of the circle  $\Gamma$ .



**Solution:** Let  $O$  be the centre of  $\Gamma$ . By symmetry  $O$  is on the perpendicular bisector of  $AB$ . Draw  $OE \perp AB$ . Then  $BE = AB/2 = 1/2$ . If  $r$  is the radius of  $\Gamma$ , we see that  $OB = 1 - r$ , and  $OE = r$ . Using Pythagoras' theorem

$$(1 - r)^2 = r^2 + \left(\frac{1}{2}\right)^2.$$

Simplification gives  $r = 3/8$ .

2. Let  $a, b, c$  be positive integers such that  $a$  divides  $b^4$ ,  $b$  divides  $c^4$  and  $c$  divides  $a^4$ . Prove that  $abc$  divides  $(a + b + c)^{21}$ .

**Solution:** If a prime  $p$  divides  $a$ , then  $p \mid b^4$  and hence  $p \mid b$ . This implies that  $p \mid c^4$  and hence  $p \mid c$ . Thus every prime dividing  $a$  also divides  $b$  and  $c$ . By symmetry, this is true for  $b$  and  $c$  as well. We conclude that  $a, b, c$  have the same set of prime divisors.

Let  $p^x \parallel a$ ,  $p^y \parallel b$  and  $p^z \parallel c$ . (Here we write  $p^x \parallel a$  to mean  $p^x \mid a$  and  $p^{x+1} \nmid a$ .) We may assume  $\min\{x, y, z\} = x$ . Now  $b \mid c^4$  implies that  $y \leq 4z$ ;  $c \mid a^4$  implies that  $z \leq 4x$ . We obtain

$$y \leq 4z \leq 16x.$$

Thus  $x + y + z \leq x + 4x + 16x = 21x$ . Hence the maximum power of  $p$  that divides  $abc$  is  $x + y + z \leq 21x$ . Since  $x$  is the minimum among  $x, y, z$ ,  $p^x$  divides  $a, b, c$ . Hence  $p^x$  divides  $a + b + c$ . This implies that  $p^{21x}$  divides  $(a + b + c)^{21}$ . Since  $x + y + z \leq 21x$ , it follows that  $p^{x+y+z}$  divides  $(a + b + c)^{21}$ . This is true of any prime  $p$  dividing  $a, b, c$ . Hence  $abc$  divides  $(a + b + c)^{21}$ .

3. Let  $a$  and  $b$  be positive real numbers such that  $a + b = 1$ . Prove that

$$a^a b^b + a^b b^a \leq 1.$$

**Solution:** Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$

Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - a^b)(b^b - b^a)$$

Now if  $a \leq b$ , then  $a^a \leq b^a$  and  $a^b \leq b^b$ . If  $a \geq b$ , then  $a^a \geq b^a$  and  $a^b \geq b^b$ . Hence the product is nonnegative for all positive  $a$  and  $b$ . It follows that

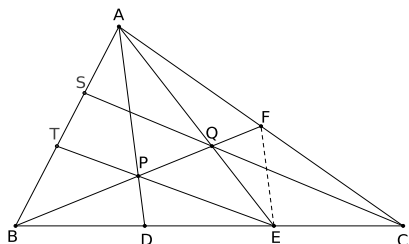
$$a^a b^b + a^b b^a \leq 1.$$

4. Let  $X = \{1, 2, 3, \dots, 12\}$ . Find the the number of pairs  $\{A, B\}$  such that  $A \subseteq X$ ,  $B \subseteq X$ ,  $A \neq B$  and  $A \cap B = \{2, 3, 5, 7, 8\}$ .

**Solution:** Let  $A \cup B = Y$ ,  $B \setminus A = M$ ,  $A \setminus B = N$  and  $X \setminus Y = L$ . Then  $X$  is the disjoint union of  $M, N, L$  and  $A \cap B$ . Now  $A \cap B = \{2, 3, 5, 7, 8\}$  is fixed. The remaining seven elements  $1, 4, 6, 9, 10, 11, 12$  can be distributed in any of the remaining sets  $M, N, L$ .

This can be done in  $3^7$  ways. Of these if all the elements are in the set  $L$ , then  $A = B = \{2, 3, 5, 7, 8\}$  and this case has to be omitted. Hence the total number of pairs  $\{A, B\}$  such that  $A \subseteq X$ ,  $B \subseteq X$ ,  $A \neq B$  and  $A \cap B = \{2, 3, 5, 7, 8\}$  is  $3^7 - 1$ .

5. Let  $ABC$  be a triangle. Let  $D, E$  be points on the segment  $BC$  such that  $BD = DE = EC$ . Let  $F$  be the mid-point of  $AC$ . Let  $BF$  intersect  $AD$  in  $P$  and  $AE$  in  $Q$  respectively. Determine  $BP/PQ$ .



**Solution:** Let  $D$  be the mid-point of  $BE$ . Join  $AD$  and let it intersect  $BF$  in  $P$ . Extend  $CQ$  and  $EP$  to meet  $AB$  in  $S$  and  $T$  respectively. Now

$$\frac{BS}{SA} = \frac{[BQC]}{[AQC]} = \frac{[BQC]/[AQB]}{[AQC]/[AQB]} = \frac{CF/FA}{EC/BE} = \frac{1}{1/2} = 2.$$

Similarly,

$$\frac{AQ}{QE} = \frac{[ABQ]}{[EBQ]} = \frac{[ACQ]}{[ECQ]} = \frac{[ABQ] + [ACQ]}{[BCQ]} = \frac{[ABQ]}{[BCQ]} + \frac{[ACQ]}{[BCQ]} = \frac{AF}{FC} + \frac{AS}{SB} = 1 + \frac{1}{2} = \frac{3}{2}.$$

And

$$\frac{AT}{TB} = \frac{[APE]}{[BPE]} = \frac{[APE]}{[APB]} \cdot \frac{[APB]}{[BPE]} = \frac{DE}{DB} \cdot \frac{AQ}{QE} = 1 \cdot \frac{3}{2} = \frac{3}{2}.$$

Finally,

$$\frac{BP}{PQ} = \frac{[BPE]}{[QPE]} = \frac{[BPA]}{[APE]} = \frac{[BPE] + [BPA]}{[APE]} = \frac{[BPE]}{[APE]} + \frac{[BPA]}{[APE]} = \frac{BT}{TA} + \frac{BD}{DE} = \frac{2}{3} + 1 = \frac{5}{3}.$$

(Note:  $BS/SA$ ,  $AT/TB$  can also be obtained using Ceva's theorem. A solution can also be obtained using coordinate geometry.)

6. Show that for all real numbers  $x, y, z$  such that  $x + y + z = 0$  and  $xy + yz + zx = -3$ , the expression  $x^3y + y^3z + z^3x$  is a constant.

**Solution:** Consider the equation whose roots are  $x, y, z$ :

$$(t - x)(t - y)(t - z) = 0.$$

This gives  $t^3 - 3t - \lambda = 0$ , where  $\lambda = xyz$ . Since  $x, y, z$  are roots of this equation, we have

$$x^3 - 3x - \lambda = 0, \quad y^3 - 3y - \lambda = 0, \quad z^3 - 3z - \lambda = 0.$$

Multiplying the first by  $y$ , the second by  $z$  and the third by  $x$ , we obtain

$$\begin{aligned} x^3y - 3xy - \lambda y &= 0, \\ y^3z - 3yz - \lambda z &= 0, \\ z^3x - 3zx - \lambda x &= 0. \end{aligned}$$

Adding we obtain

$$x^3y + y^3z + z^3x - 3(xy + yz + zx) - \lambda(x + y + z) = 0.$$

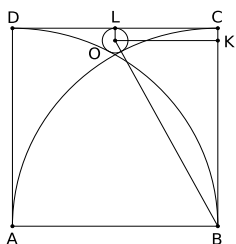
This simplifies to

$$x^3y + y^3z + z^3x = -9.$$

(Here one may also solve for  $y$  and  $z$  in terms of  $x$  and substitute these values in  $x^3y + y^3z + z^3x$  to get  $-9$ .)

## Problems and Solutions: CRMO-2012, Paper 3

1. Let  $ABCD$  be a unit square. Draw a quadrant of a circle with  $A$  as centre and  $B, D$  as end points of the arc. Similarly, draw a quadrant of a circle with  $B$  as centre and  $A, C$  as end points of the arc. Inscribe a circle  $\Gamma$  touching the arcs  $AC$  and  $BD$  both externally and also touching the side  $CD$ . Find the radius of the circle  $\Gamma$ .



**Solution:** Let  $O$  be the centre of  $\Gamma$ . By symmetry  $O$  is on the perpendicular bisector of  $CD$ . Draw  $OL \perp CD$  and  $OK \perp BC$ . Then  $OK = CL = CD/2 = 1/2$ . If  $r$  is the radius of  $\Gamma$ , we see that  $BK = 1 - r$ , and  $OE = r$ . Using Pythagoras' theorem

$$(1 + r)^2 = (1 - r)^2 + \left(\frac{1}{2}\right)^2.$$

Simplification gives  $r = 1/16$ .

2. Let  $a, b, c$  be positive integers such that  $a$  divides  $b^5$ ,  $b$  divides  $c^5$  and  $c$  divides  $a^5$ . Prove that  $abc$  divides  $(a + b + c)^{31}$ .

**Solution:** If a prime  $p$  divides  $a$ , then  $p \mid b^5$  and hence  $p \mid b$ . This implies that  $p \mid c^4$  and hence  $p \mid c$ . Thus every prime dividing  $a$  also divides  $b$  and  $c$ . By symmetry, this is true for  $b$  and  $c$  as well. We conclude that  $a, b, c$  have the same set of prime divisors.

Let  $p^x \parallel a$ ,  $p^y \parallel b$  and  $p^z \parallel c$ . (Here we write  $p^x \parallel a$  to mean  $p^x \mid a$  and  $p^{x+1} \nmid a$ .) We may assume  $\min\{x, y, z\} = x$ . Now  $b \mid c^5$  implies that  $y \leq 5z$ ;  $c \mid a^5$  implies that  $z \leq 5x$ . We obtain

$$y \leq 5z \leq 25x.$$

Thus  $x + y + z \leq x + 5x + 25x = 31x$ . Hence the maximum power of  $p$  that divides  $abc$  is  $x + y + z \leq 31x$ . Since  $x$  is the minimum among  $x, y, z$ ,  $p^x$  divides  $a, b, c$ . Hence  $p^x$  divides  $a + b + c$ . This implies that  $p^{31x}$  divides  $(a + b + c)^{21}$ . Since  $x + y + z \leq 31x$ , it follows that  $p^{x+y+z}$  divides  $(a + b + c)^{31}$ . This is true of any prime  $p$  dividing  $a, b, c$ . Hence  $abc$  divides  $(a + b + c)^{31}$ .

3. Let  $a$  and  $b$  be positive real numbers such that  $a + b = 1$ . Prove that

$$a^a b^b + a^b b^a \leq 1.$$

**Solution:** Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$

Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$

Now if  $a \leq b$ , then  $a^a \leq b^a$  and  $a^b \leq b^b$ . If  $a \geq b$ , then  $a^a \geq b^a$  and  $a^b \geq b^b$ . Hence the product is nonnegative for all positive  $a$  and  $b$ . It follows that

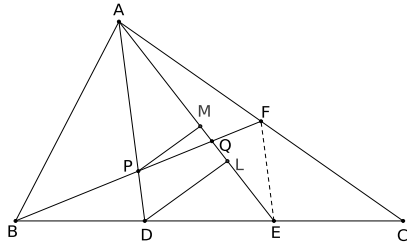
$$a^a b^b + a^b b^a \leq 1.$$

4. Let  $X = \{1, 2, 3, \dots, 10\}$ . Find the number of pairs  $\{A, B\}$  such that  $A \subseteq X$ ,  $B \subseteq X$ ,  $A \neq B$  and  $A \cap B = \{5, 7, 8\}$ .

**Solution:** Let  $A \cup B = Y$ ,  $B \setminus A = M$ ,  $A \setminus B = N$  and  $X \setminus Y = L$ . Then  $X$  is the disjoint union of  $M, N, L$  and  $A \cap B$ . Now  $A \cap B = \{5, 7, 8\}$  is fixed. The remaining seven elements  $1, 2, 3, 4, 6, 9, 10$  can be distributed in any of the remaining sets  $M$ ,

$N, L$ . This can be done in  $3^7$  ways. Of these if all the elements are in the set  $L$ , then  $A = B = \{5, 7, 8\}$  and this case has to be omitted. Hence the total number of pairs  $\{A, B\}$  such that  $A \subseteq X, B \subseteq X, A \neq B$  and  $A \cap B = \{5, 7, 8\}$  is  $3^7 - 1$ .

5. Let  $ABC$  be a triangle. Let  $D, E$  be points on the segment  $BC$  such that  $BD = DE = EC$ . Let  $F$  be the mid-point of  $AC$ . Let  $BF$  intersect  $AD$  in  $P$  and  $AE$  in  $Q$  respectively. Determine the ratio of the area of the triangle  $APQ$  to that of the quadrilateral  $PDEQ$ .



**Solution:** If we can find  $[APQ]/[ADE]$ , then we can get the required ratio as

$$\begin{aligned} \frac{[APQ]}{[PDEQ]} &= \frac{[APQ]}{[ADE] - [APQ]} \\ &= \frac{1}{([ADE]/[APQ]) - 1}. \end{aligned}$$

Now draw  $PM \perp AE$  and  $DL \perp AE$ . Observe

$$[APQ] = \frac{1}{2}AQ \cdot PM, [ADE] = \frac{1}{2}AE \cdot DL.$$

Further, since  $PM \parallel DL$ , we also get  $PM/DL = AP/AD$ . Using these we obtain

$$\frac{[APQ]}{[ADE]} = \frac{AP}{AD} \cdot \frac{AQ}{AE}.$$

We have

$$\frac{AQ}{QE} = \frac{[ABQ]}{[EBQ]} = \frac{[ACQ]}{[ECQ]} = \frac{[ABQ] + [ACQ]}{[BCQ]} = \frac{[ABQ]}{[BCQ]} + \frac{[ACQ]}{[BCQ]} = \frac{AF}{FC} + \frac{AS}{SB}.$$

However

$$\frac{BS}{SA} = \frac{[BQC]}{[AQC]} = \frac{[BQC]/[AQB]}{[AQC]/[AQB]} = \frac{CF/FA}{EC/BE} = \frac{1}{1/2} = 2.$$

Besides  $AF/FC = 1$ . We obtain

$$\frac{AQ}{QE} = \frac{AF}{FC} + \frac{AS}{SB} = 1 + \frac{1}{2} = \frac{3}{2}, \quad \frac{AE}{QE} = 1 + \frac{3}{2} = \frac{5}{2}, \quad \frac{AQ}{AE} = \frac{3}{5}.$$

Since  $EF \parallel AD$  (since  $DE/EC = AF/FC = 1$ ), we get  $AD = 2EF$ . Since  $EF \parallel PD$ , we also have  $PD/EF = BD/DE = 1/2$ . Hence  $EF = 2PD$ . Thus  $AD = 4PD$ . This gives and  $AP/PD = 3$  and  $AP/AD = 3/4$ . Thus

$$\frac{[APQ]}{[ADE]} = \frac{AP}{AD} \cdot \frac{AQ}{AE} = \frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20}.$$

Finally,

$$\frac{[APQ]}{[PDEQ]} = \frac{1}{([ADE]/[APQ]) - 1} = \frac{1}{(20/9) - 1} = \frac{9}{11}.$$

(Note:  $BS/SA$  can also be obtained using Ceva's theorem. Coordinate geometry solution can also be obtained.)

6. Find all positive integers  $n$  such that  $3^{2n} + 3n^2 + 7$  is a perfect square.

**Solution:** If  $3^{2n} + 3n^2 + 7 = b^2$  for some natural number  $b$ , then  $b^2 > 3^{2n}$  so that  $b > 3^n$ . This implies that  $b \geq 3^n + 1$ . Thus

$$3^{2n} + 3n^2 + 7 = b^2 \geq (3^n + 1)^2 = 3^{2n} + 2 \cdot 3^n + 1.$$

This shows that  $2 \cdot 3^n \leq 3n^2 + 6$ . If  $n \geq 3$ , this cannot hold. One can prove this either by induction or by direct argument:

If  $n \geq 3$ , then

$$\begin{aligned} 2 \cdot 3^n &= 2(1+2)^n = 2 \left( 1 + 2n + \binom{n}{2} 2^2 + \dots \right) > 2 + 4n + 4n^2 - 4n \\ &= 3n^2 + (n^2 + 2) \geq 3n^2 + 11 > 3n^2 + 6. \end{aligned}$$

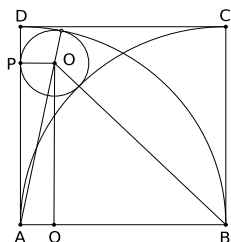
Hence  $n = 1$  or  $2$ .

If  $n = 1$ , then  $3^{2n} + 3n^2 + 7 = 19$  and this is not a perfect square. If  $n = 2$ , we obtain  $3^{2n} + 3n^2 + 7 = 81 + 12 + 7 = 100 = 10^2$ . Hence  $n = 2$  is the only solution.

————00000————

## Problems and Solutions: CRMO-2012, Paper 4

1. Let  $ABCD$  be a unit square. Draw a quadrant of a circle with  $A$  as centre and  $B, D$  as end points of the arc. Similarly, draw a quadrant of a circle with  $B$  as centre and  $A, C$  as end points of the arc. Inscribe a circle  $\Gamma$  touching the arc  $AC$  externally, the arc  $BD$  internally and also touching the side  $AD$ . Find the radius of the circle  $\Gamma$ .



**Solution:** Let  $O$  be the centre of  $\Gamma$  and  $r$  its radius. Draw  $OP \perp AD$  and  $OQ \perp AB$ . Then  $OP = r$ ,  $OQ^2 = OA^2 - r^2 = (1 - r)^2 - r^2 = 1 - 2r$ . We also have  $OB = 1 + r$  and  $BQ = 1 - r$ . Using Pythagoras' theorem we get

$$(1 + r)^2 = (1 - r)^2 + 1 - 2r.$$

Simplification gives  $r = 1/6$ .

2. Let  $a, b, c$  be positive integers such that  $a$  divides  $b^2$ ,  $b$  divides  $c^2$  and  $c$  divides  $a^2$ . Prove that  $abc$  divides  $(a + b + c)^7$ .

**Solution:** If a prime  $p$  divides  $a$ , then  $p \mid b^2$  and hence  $p \mid b$ . This implies that  $p \mid c^2$  and hence  $p \mid c$ . Thus every prime dividing  $a$  also divides  $b$  and  $c$ . By symmetry, this is true for  $b$  and  $c$  as well. We conclude that  $a, b, c$  have the same set of prime divisors.

Let  $p^x \parallel a$ ,  $p^y \parallel b$  and  $p^z \parallel c$ . (Here we write  $p^x \parallel a$  to mean  $p^x \mid a$  and  $p^{x+1} \nmid a$ .) We may assume  $\min\{x, y, z\} = x$ . Now  $b \mid c^2$  implies that  $y \leq 2z$ ;  $c \mid a^2$  implies that  $z \leq 2x$ . We obtain

$$y \leq 2z \leq 4x.$$

Thus  $x + y + z \leq x + 2x + 4x = 7x$ . Hence the maximum power of  $p$  that divides  $abc$  is  $x + y + z \leq 7x$ . Since  $x$  is the minimum among  $x, y, z$ ,  $p^x$  divides  $a, b, c$ . Hence  $p^x$  divides  $a + b + c$ . This implies that  $p^{7x}$  divides  $(a + b + c)^7$ . Since  $x + y + z \leq 7x$ , it follows that  $p^{x+y+z}$  divides  $(a + b + c)^7$ . This is true of any prime  $p$  dividing  $a, b, c$ . Hence  $abc$  divides  $(a + b + c)^7$ .

3. Let  $a$  and  $b$  be positive real numbers such that  $a + b = 1$ . Prove that

$$a^a b^b + a^b b^a \leq 1.$$

**Solution:** Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$

Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$

Now if  $a \leq b$ , then  $a^a \leq b^a$  and  $a^b \leq b^b$ . If  $a \geq b$ , then  $a^a \geq b^a$  and  $a^b \geq b^b$ . Hence the product is nonnegative for all positive  $a$  and  $b$ . It follows that

$$a^a b^b + a^b b^a \leq 1.$$

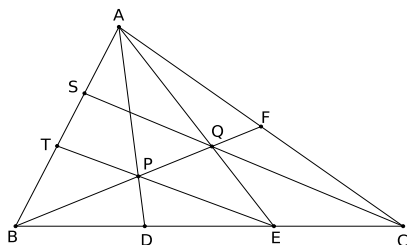
4. Let  $X = \{1, 2, 3, \dots, 11\}$ . Find the number of pairs  $\{A, B\}$  such that  $A \subseteq X$ ,  $B \subseteq X$ ,  $A \neq B$  and  $A \cap B = \{4, 5, 7, 8, 9, 10\}$ .

**Solution:** Let  $A \cup B = Y$ ,  $B \setminus A = M$ ,  $A \setminus B = N$  and  $X \setminus Y = L$ . Then  $X$  is the disjoint union of  $M$ ,  $N$ ,  $L$  and  $A \cap B$ . Now  $A \cap B = \{4, 5, 7, 8, 9, 10\}$  is fixed. The remaining 5 elements  $1, 2, 3, 6, 11$  can be distributed in any of the remaining sets  $M$ ,

$N, L$ . This can be done in  $3^5$  ways. Of these if all the elements are in the set  $L$ , then  $A = B = \{4, 5, 7, 8, 9, 10\}$  and this case has to be omitted. Hence the total number of pairs  $\{A, B\}$  such that  $A \subseteq X, B \subseteq X, A \neq B$  and  $A \cap B = \{4, 5, 7, 8, 9, 10\}$  is  $3^5 - 1$ .

5. Let  $ABC$  be a triangle. Let  $E$  be a point on the segment  $BC$  such that  $BE = 2EC$ . Let  $F$  be the mid-point of  $AC$ . Let  $BF$  intersect  $AE$  in  $Q$ . Determine  $BQ/QF$ .

**Solution:** Let  $CQ$  and  $ET$  meet  $AB$  in  $S$  and  $T$  respectively. We have



$$\frac{[SBC]}{[ASC]} = \frac{BS}{SA} = \frac{[SBQ]}{[ASQ]}.$$

Using componendo by dividendo, we obtain

$$\frac{BS}{SA} = \frac{[SBC] - [SBQ]}{[ASC] - [ASQ]} = \frac{[BQC]}{[AQC]}.$$

Similarly, We can prove

$$\frac{BE}{EC} = \frac{[BQA]}{[CQA]}, \quad \frac{CF}{FA} = \frac{[CQB]}{[AQB]}.$$

But  $BD = DE = EC$  implies that  $BE/EC = 2$ ;  $CF = FA$  gives  $CF/FA = 1$ . Thus

$$\frac{BS}{SA} = \frac{[BQC]}{[AQC]} = \frac{[BQC]/[AQB]}{[AQC]/[AQB]} = \frac{CF/FA}{EC/BE} = \frac{1}{1/2} = 2.$$

Now

$$\frac{BQ}{QF} = \frac{[BQC]}{[FQC]} = \frac{[BQA]}{[FQA]} = \frac{[BQC] + [BQA]}{[FQC] + [FQA]} = \frac{[BQC] + [BQA]}{[AQC]}.$$

This gives

$$\frac{BQ}{QF} = \frac{[BQC] + [BQA]}{[AQC]} = \frac{[BQC]}{[AQC]} + \frac{[BQA]}{[AQC]} = \frac{BS}{SA} + \frac{BE}{EC} = 2 + 2 = 4.$$

(Note:  $BS/SA$  can also be obtained using Ceva's theorem. One can also obtain the result by coordinate geometry.)

6. Solve the system of equations for positive real numbers:

$$\frac{1}{xy} = \frac{x}{z} + 1, \quad \frac{1}{yz} = \frac{y}{x} + 1, \quad \frac{1}{zx} = \frac{z}{y} + 1.$$

**Solution:** The given system reduces to

$$z = x^2y + xyz, \quad x = y^2z + xyz, \quad y = z^2x + xyz.$$

Hence

$$z - x^2y = x - y^2z = y - z^2x.$$

If  $x = y$ , then  $y^2z = z^2x$  and hence  $x^2z = z^2x$ . This implies that  $z = x = y$ . Similarly,  $x = z$  implies that  $x = z = y$ . Hence if any two of  $x, y, z$  are equal, then all are equal.

Suppose no two of  $x, y, z$  are equal. We may take  $x$  is the largest among  $x, y, z$  so that  $x > y$  and  $x > z$ . Here we have two possibilities:  $y > z$  and  $z > y$ .

Suppose  $x > y > z$ . Now  $z - x^2y = x - y^2z = y - z^2x$  shows that

$$y^2z > z^2x > x^2y.$$

But  $y^2z > z^2x$  and  $z^2x > x^2y$  give  $y^2 > zx$  and  $z^2 > xy$ . Hence  
 $(y^2)(z^2) > (zx)(xy)$ .

This gives  $yz > x^2$ . Thus  $x^3 < xyz = (xz)y < (y^2)y = y^3$ . This forces  $x < y$  contradicting  $x > y$ .

Similarly, we arrive at a contradiction if  $x > z > y$ . The only possibility is  $x = y = z$ .

For  $x = y = z$ , we get only one equation  $x^2 = 1/2$ . Since  $x > 0$ ,  $x = 1/\sqrt{2} = y = z$ .

————-0000————-

1. Let  $ABC$  be an acute-angled triangle. The circle  $\Gamma$  with  $BC$  as diameter intersects  $AB$  and  $AC$  again at  $P$  and  $Q$ , respectively. Determine  $\angle BAC$  given that the orthocentre of triangle  $APQ$  lies on  $\Gamma$ .
2. Let  $f(x) = x^3 + ax^2 + bx + c$  and  $g(x) = x^3 + bx^2 + cx + a$ , where  $a, b, c$  are integers with  $c \neq 0$ . Suppose that the following conditions hold:
  - (a)  $f(1) = 0$ ;
  - (b) the roots of  $g(x) = 0$  are the squares of the roots of  $f(x) = 0$ .

Find the value of  $a^{2013} + b^{2013} + c^{2013}$ .

3. Find all primes  $p$  and  $q$  such that  $p$  divides  $q^2 - 4$  and  $q$  divides  $p^2 - 1$ .
4. Find the number of 10-tuples  $(a_1, a_2, \dots, a_{10})$  of integers such that  $|a_i| \leq 1$  and

$$a_1^2 + a_2^2 + a_3^2 + \dots + a_{10}^2 - a_1a_2 - a_2a_3 - a_3a_4 - \dots - a_9a_{10} - a_{10}a_1 = 2.$$

5. Let  $ABC$  be a triangle with  $\angle A = 90^\circ$  and  $AB = AC$ . Let  $D$  and  $E$  be points on the segment  $BC$  such that  $BD : DE : EC = 3 : 5 : 4$ . Prove that  $\angle DAE = 45^\circ$ .
6. Suppose that  $m$  and  $n$  are integers such that both the quadratic equations  $x^2 + mx - n = 0$  and  $x^2 - mx + n = 0$  have integer roots. Prove that  $n$  is divisible by 6.

1. Let  $ABC$  be an acute angled triangle. The circle  $\Gamma$  with  $BC$  as diameter intersects  $AB$  and  $AC$  again at  $P$  and  $Q$ , respectively. Determine  $\angle BAC$  given that the orthocenter of triangle  $APQ$  lies on  $\Gamma$ .

**Solution.** Let  $K$  denote the orthocenter of triangle  $APQ$ . Since triangles  $ABC$  and  $AQP$  are similar it follows that  $K$  lies in the interior of triangle  $APQ$ .

Note that  $\angle KPA = \angle KQA = 90^\circ - \angle A$ . Since  $BPKQ$  is a cyclic quadrilateral it follows that  $\angle BQK = 180^\circ - \angle BPK = 90^\circ - \angle A$ , while on the other hand  $\angle BQK = \angle BQA - \angle KQA = \angle A$  since  $BQ$  is perpendicular to  $AC$ . This shows that  $90^\circ - \angle A = \angle A$ , so  $\angle A = 45^\circ$ .  $\square$

2. Let  $f(x) = x^3 + ax^2 + bx + c$  and  $g(x) = x^3 + bx^2 + cx + a$ , where  $a, b, c$  are integers with  $c \neq 0$ . Suppose that the following conditions hold:

- (a)  $f(1) = 0$ ;  
 (b) the roots of  $g(x)$  are squares of the roots of  $f(x)$ .

Find the value of  $a^{2013} + b^{2013} + c^{2013}$ .

**Solution.** Note that  $g(1) = f(1) = 0$ , so 1 is a root of both  $f(x)$  and  $g(x)$ . Let  $p$  and  $q$  be the other two roots of  $f(x)$ , so  $p^2$  and  $q^2$  are the other two roots of  $g(x)$ . We then get  $pq = -c$  and  $p^2q^2 = -a$ , so  $a = -c^2$ . Also,  $(-a)^2 = (p+q+1)^2 = p^2 + q^2 + 1 + 2(pq + p + q) = -b + 2b = b$ . Therefore  $b = c^4$ . Since  $f(1) = 0$  we therefore get  $1 + c - c^2 + c^4 = 0$ . Factorising, we get  $(c+1)(c^3 - c^2 + 1) = 0$ . Note that  $c^3 - c^2 + 1 = 0$  has no integer root and hence  $c = -1, b = 1, a = -1$ . Therefore  $a^{2013} + b^{2013} + c^{2013} = -1$ .  $\square$

3. Find all primes  $p$  and  $q$  such that  $p$  divides  $q^2 - 4$  and  $q$  divides  $p^2 - 1$ .

**Solution.** Suppose that  $p \leq q$ . Since  $q$  divides  $(p-1)(p+1)$  and  $q > p-1$  it follows that  $q$  divides  $p+1$  and hence  $q = p+1$ . Therefore  $p = 2$  and  $q = 3$ .

On the other hand, if  $p > q$  then  $p$  divides  $(q-2)(q+2)$  implies that  $p$  divides  $q+2$  or  $q-2 = 0$ . This gives either  $p = q+2$  or  $q = 2$ . In the former case it follows that  $q$  divides  $(q+2)^2 - 1$ , so  $q$  divides 3. This gives the solutions  $p > 2, q = 2$  and  $(p, q) = (5, 3)$ .  $\square$

4. Find the number of 10-tuples  $(a_1, a_2, \dots, a_{10})$  of integers such that  $|a_1| \leq 1$  and

$$a_1^2 + a_2^2 + a_3^2 + \dots + a_{10}^2 - a_1a_2 - a_2a_3 - a_3a_4 - \dots - a_9a_{10} - a_{10}a_1 = 2.$$

**Solution.** Let  $a_{11} = a_1$ . Multiplying the given equation by 2 we get

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 + \dots + (a_{10} - a_1)^2 = 4.$$

Note that if  $a_i - a_{i+1} = \pm 2$  for some  $i = 1, \dots, 10$ , then  $a_j - a_{j+1} = 0$  for all  $j \neq i$  which contradicts the equality  $\sum_{i=1}^{10} (a_i - a_{i+1}) = 0$ . Therefore  $a_i - a_{i+1} = 1$  for exactly two values of  $i$  in  $\{1, 2, \dots, 10\}$ ,  $a_i - a_{i+1} = -1$  for two other values of  $i$  and  $a_i - a_{i+1} = 0$  for all other values of  $i$ . There are  $\binom{10}{2} \times \binom{8}{2} = 45 \times 28$  possible ways of choosing these values. Note that  $a_1 = -1, 0$  or  $1$ , so in total there are  $3 \times 45 \times 28$  possible integer solutions to the given equation.  $\square$

5. Let  $ABC$  be a triangle with  $\angle A = 90^\circ$  and  $AB = AC$ . Let  $D$  and  $E$  be points on the segment  $BC$  such that  $BD : DE : EC = 3 : 5 : 4$ . Prove that  $\angle DAE = 45^\circ$ .

**Solution.** Rotating the configuration about  $A$  by  $90^\circ$ , the point  $B$  goes to the point  $C$ . Let  $P$  denote the image of the point  $D$  under this rotation. Then  $CP = BD$  and  $\angle ACP = \angle ABC = 45^\circ$ , so  $ECP$  is a right-angled triangle with  $CE : CP = 4 : 3$ . Hence  $PE = ED$ . It follows that  $ADEP$  is a kite with  $AP = AD$  and  $PE = ED$ . Therefore  $AE$  is the angular bisector of  $\angle PAD$ . This implies that  $\angle DAE = \angle PAD/2 = 45^\circ$ .  $\square$

6. Suppose that  $m$  and  $n$  are integers such that both the quadratic equations  $x^2 + mx - n = 0$  and  $x^2 - mx + n = 0$  have integer roots. Prove that  $n$  is divisible by 6.

**Solution.** Let  $a$  be an integer. If  $a$  is not divisible by 3 then  $a^2 \equiv 1 \pmod{3}$ , i.e., 3 divides  $a^2 - 1$ , and if  $a$  is odd then  $a^2 \equiv 1 \pmod{8}$ , i.e., 8 divides  $a^2 - 1$ .

Note that the discriminants of the two quadratic polynomials are both squares of integers. Let  $a$  and  $b$  be integers such that  $m^2 - 4n = a^2$  and  $m^2 + 4n = b^2$ . Therefore  $8n = b^2 - a^2$  and  $2m^2 = a^2 + b^2$ . If 3 divides  $m$  then 3 divides both  $a$  and  $b$ , so 3 divides  $n$ . On the other hand if 3 does not divide  $m$  then 3 does not divide  $a$  or  $b$ . Therefore 3 divides  $b^2 - a^2$  and hence 3 divides  $n$ .

If  $m$  is odd, then so is  $a$ , and therefore  $4n = m^2 - a^2$  is divisible by 8, so  $n$  is even. On the other hand, if  $m$  is even then both  $a$  and  $b$  are even. Further  $(m/2)^2 - n = (a/2)^2$  and  $(m/2)^2 + n = (b/2)^2$ , so  $(b - a)/2$  is even. In particular,  $n = (b^2 - a^2)/4$  is even.  $\square$

————— ★ —————

1. Prove that there do not exist natural numbers  $x$  and  $y$ , with  $x > 1$ , such that

$$\frac{x^7 - 1}{x - 1} = y^5 + 1.$$

2. In a triangle  $ABC$ ,  $AD$  is the altitude from  $A$ , and  $H$  is the orthocentre. Let  $K$  be the centre of the circle passing through  $D$  and tangent to  $BH$  at  $H$ . Prove that the line  $DK$  bisects  $AC$ .

3. Consider the expression

$$2013^2 + 2014^2 + 2015^2 + \cdots + n^2.$$

Prove that there exists a natural number  $n > 2013$  for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

4. Let  $ABC$  be a triangle with  $\angle A = 90^\circ$  and  $AB = AC$ . Let  $D$  and  $E$  be points on the segment  $BC$  such that  $BD : DE : EC = 1 : 2 : \sqrt{3}$ . Prove that  $\angle DAE = 45^\circ$ .
5. Let  $n \geq 3$  be a natural number and let  $P$  be a polygon with  $n$  sides. Let  $a_1, a_2, \dots, a_n$  be the lengths of the sides of  $P$  and let  $p$  be its perimeter. Prove that

$$\frac{a_1}{p - a_1} + \frac{a_2}{p - a_2} + \cdots + \frac{a_n}{p - a_n} < 2.$$

6. For a natural number  $n$ , let  $T(n)$  denote the number of ways we can place  $n$  objects of weights  $1, 2, \dots, n$  on a balance such that the sum of the weights in each pan is the same. Prove that  $T(100) > T(99)$ .

———— ★★ ————

1. Prove that there do not exist natural numbers  $x$  and  $y$ , with  $x > 1$ , such that

$$\frac{x^7 - 1}{x - 1} = y^5 + 1.$$

**Solution.** Simple factorisation gives  $y^5 = x(x^3 + 1)(x^2 + x + 1)$ . The three factors on the right are mutually coprime and hence they all have to be fifth powers. In particular,  $x = r^5$  for some integer  $r$ . This implies  $x^3 + 1 = r^{15} + 1$ , which is not a fifth power unless  $r = -1$  or  $r = 0$ . This implies there are no solutions to the given equation.  $\square$

2. In a triangle  $ABC$ ,  $AD$  is the altitude from  $A$ , and  $H$  is the orthocentre. Let  $K$  be the centre of the circle passing through  $D$  and tangent to  $BH$  at  $H$ . Prove that the line  $DK$  bisects  $AC$ .

**Solution.** Note that  $\angle KHB = 90^\circ$ . Therefore  $\angle KDA = \angle KHD = 90^\circ - \angle BHD = \angle HBD = \angle HAC$ . On the other hand, if  $M$  is the midpoint of  $AC$  then it is the circumcenter of triangle  $ADC$  and therefore  $\angle MDA = \angle MAD$ . This proves that  $D, K, M$  are collinear and hence  $DK$  bisects  $AC$ .  $\square$

3. Consider the expression

$$2013^2 + 2014^2 + 2015^2 + \cdots + n^2.$$

Prove that there exists a natural number  $n > 2013$  for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

**Solution.** For any integer  $k$  we have

$$-k^2 + (k+1)^2 + (k+2)^2 - (k+3)^2 = -4.$$

Note that  $9999 - (2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2) = -4m$  for some positive integer  $m$ . Therefore, it follows that

$$\begin{aligned} 9999 = & (2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2) \\ & + \sum_{r=1}^m (-(4r+2014)^2 + (4r+2015)^2 + (4r+2016)^2 - (4r+2017)^2). \end{aligned}$$

$\square$

4. Let  $ABC$  be a triangle with  $\angle A = 90^\circ$  and  $AB = AC$ . Let  $D$  and  $E$  be points on the segment  $BC$  such that  $BD : DE : EC = 1 : 2 : \sqrt{3}$ . Prove that  $\angle DAE = 45^\circ$ .

**Solution.** Rotating the configuration about  $A$  by  $90^\circ$ , the point  $B$  goes to the point  $C$ . Let  $P$  denote the image of the point  $D$  under this rotation. Then  $CP = BD$  and  $\angle ACP = \angle ABC = 45^\circ$ , so  $ECP$  is a right-angled triangle with  $CE : CP = \sqrt{3} : 1$ . Hence  $PE = ED$ . It follows that  $ADEP$  is a kite with  $AP = AD$  and  $PE = ED$ . Therefore  $AE$  is the angular bisector of  $\angle PAD$ . This implies that  $\angle DAE = \angle PAD/2 = 45^\circ$ .  $\square$

5. Let  $n \geq 3$  be a natural number and let  $P$  be a polygon with  $n$  sides. Let  $a_1, a_2, \dots, a_n$  be the lengths of the sides of  $P$  and let  $p$  be its perimeter. Prove that

$$\frac{a_1}{p - a_1} + \frac{a_2}{p - a_2} + \dots + \frac{a_n}{p - a_n} < 2.$$

**Solution.** If  $r$  and  $s$  are positive real numbers such that  $r < s$  then  $r/s < (r + x)/(s + x)$  for any positive real  $x$ . Note that, by triangle inequality,  $a_i < p - a_i$ , so

$$\frac{a_i}{p - a_i} < \frac{2a_i}{p},$$

for all  $i = 1, 2, \dots, n$ . Summing this inequality over  $i$  we get the desired inequality.  $\square$

6. For a natural number  $n$ , let  $T(n)$  denote the number of ways we can place  $n$  objects of weights  $1, 2, \dots, n$  on a balance such that the sum of the weights in each pan is the same. Prove that  $T(100) > T(99)$ .

**Solution.** Let  $\mathcal{S}(n)$  denote the collection of subsets  $A$  of  $X(n) = \{1, 2, \dots, n\}$  such that the sum of the elements of  $A$  equals  $n(n+1)/4$ . Then the given inequality is equivalent to  $|\mathcal{S}(100)| > |\mathcal{S}(99)|$ . We shall give a map  $f : \mathcal{S}(99) \rightarrow \mathcal{S}(100)$  which is one-to-one but not onto. Note that this will prove the required inequality.

Suppose that  $A$  is an element of  $\mathcal{S}(99)$ . If  $50 \in A$  then define  $f(A) = (A \setminus \{50\}) \cup \{100\}$ . Otherwise, define  $f(A) = A \cup \{50\}$ . If  $A$  and  $B$  are elements of  $\mathcal{S}(99)$  such that  $f(A) = f(B)$  then either 50 belongs to both these sets or neither of these sets. In either of the cases we have  $A = B$ . Therefore  $f$  is a one-to-one function.

Note that every element in the range of  $f$  contains exactly one of 50 and 100. Let  $B_i = \{i, 101 - i\}$ . Then  $B_1 \cup B_2 \cup \dots \cup B_{24} \cup B_{50}$  is an element of  $\mathcal{S}(100)$ . Clearly, this is not in the range of  $f$ . Thus  $f$  is not an onto function.  $\square$

————— ★★ —————

1. Find the number of eight-digit numbers the sum of whose digits is 4.
2. Find all 4-tuples  $(a, b, c, d)$  of natural numbers with  $a \leq b \leq c$  and  $a! + b! + c! = 3^d$ .
3. In an acute-angled triangle  $ABC$  with  $AB < AC$ , the circle  $\Gamma$  touches  $AB$  at  $B$  and passes through  $C$  intersecting  $AC$  again at  $D$ . Prove that the orthocentre of triangle  $ABD$  lies on  $\Gamma$  if and only if it lies on the perpendicular bisector of  $BC$ .
4. A polynomial is called a *Fermat polynomial* if it can be written as the sum of the squares of two polynomials with integer coefficients. Suppose that  $f(x)$  is a Fermat polynomial such that  $f(0) = 1000$ . Prove that  $f(x) + 2x$  is not a Fermat polynomial.
5. Let  $ABC$  be a triangle which is not right-angled. Define a sequence of triangles  $A_i B_i C_i$ , with  $i \geq 0$ , as follows:  $A_0 B_0 C_0$  is the triangle  $ABC$ ; and, for  $i \geq 0$ ,  $A_{i+1}, B_{i+1}, C_{i+1}$  are the reflections of the orthocentre of triangle  $A_i B_i C_i$  in the sides  $B_i C_i, C_i A_i, A_i B_i$ , respectively. Assume that  $\angle A_m = \angle A_n$  for some distinct natural numbers  $m, n$ . Prove that  $\angle A = 60^\circ$ .
6. Let  $n \geq 4$  be a natural number. Let  $A_1 A_2 \cdots A_n$  be a regular polygon and  $X = \{1, 2, \dots, n\}$ . A subset  $\{i_1, i_2, \dots, i_k\}$  of  $X$ , with  $k \geq 3$  and  $i_1 < i_2 < \cdots < i_k$ , is called a *good subset* if the angles of the polygon  $A_{i_1} A_{i_2} \cdots A_{i_k}$ , when arranged in the increasing order, are in an arithmetic progression. If  $n$  is a prime, show that a **proper** good subset of  $X$  contains exactly four elements.

————— ★ ★ ★ —————

1. Let  $\Gamma$  be a circle with centre  $O$ . Let  $\Lambda$  be another circle passing through  $O$  and intersecting  $\Gamma$  at points  $A$  and  $B$ . A diameter  $CD$  of  $\Gamma$  intersects  $\Lambda$  at a point  $P$  different from  $O$ . Prove that

$$\angle APC = \angle BPD.$$

2. Determine the smallest prime that does not divide any five-digit number whose digits are in a strictly increasing order.
3. Given real numbers  $a, b, c, d, e > 1$  prove that

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20.$$

4. Let  $x$  be a non-zero real number such that  $x^4 + \frac{1}{x^4}$  and  $x^5 + \frac{1}{x^5}$  are both rational numbers. Prove that  $x + \frac{1}{x}$  is a rational number.
5. In a triangle  $ABC$ , let  $H$  denote its orthocentre. Let  $P$  be the reflection of  $A$  with respect to  $BC$ . The circumcircle of triangle  $ABP$  intersects the line  $BH$  again at  $Q$ , and the circumcircle of triangle  $ACP$  intersects the line  $CH$  again at  $R$ . Prove that  $H$  is the incentre of triangle  $PQR$ .
6. Suppose that the vertices of a regular polygon of 20 sides are coloured with three colours – red, blue and green – such that there are exactly three red vertices. Prove that there are three vertices  $A, B, C$  of the polygon having the same colour such that triangle  $ABC$  is isosceles.

————— ★ ★ ★ ★ —————

1. Let  $\Gamma$  be a circle with centre  $O$ . Let  $\Lambda$  be another circle passing through  $O$  and intersecting  $\Gamma$  at points  $A$  and  $B$ . A diameter  $CD$  of  $\Gamma$  intersects  $\Lambda$  at a point  $P$  different from  $O$ . Prove that

$$\angle APC = \angle BPD.$$

**Solution.** Suppose that  $A'$  is a point on  $\Lambda$  such that  $\angle A'PC = \angle BPD$ . Then the segments  $OA'$  and  $OB$  subtends same angle in the respective minor arcs, so  $OA' = OB$ . This shows that  $A$  lies on  $\Gamma$  and hence  $A' = A$ . This proves that  $\angle APC = \angle BPD$ .  $\square$

2. Determine the smallest prime that does not divide any five-digit number whose digits are in a strictly increasing order.

**Solution.** Note that 12346 is even, 3 and 5 divide 12345, and 7 divides 12348. Consider a 5 digit number  $n = abcde$  with  $0 < a < b < c < d < e < 10$ . Let  $S = (a + c + e) - (b + d)$ . Then  $S = a + (c - b) + (e - d) > a > 0$  and  $S = e - (d - c) - (b - a) < e \leq 10$ , so  $S$  is not divisible by 11 and hence  $n$  is not divisible by 11. Thus 11 is the smallest prime that does not divide any five-digit number whose digits are in a strictly increasing order.  $\square$

3. Given real numbers  $a, b, c, d, e > 1$  prove that

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20.$$

**Solution.** Note that  $(a-2)^2 \geq 0$  and hence  $a^2 \geq 4(a-1)$ . Since  $a > 1$  we have  $\frac{a^2}{a-1} \geq 4$ . By applying AM-GM inequality we get

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 5 \sqrt[5]{\frac{a^2 b^2 c^2 d^2 e^2}{(a-1)(b-1)(c-1)(d-1)(e-1)}} \geq 20.$$

$\square$

4. Let  $x$  be a non-zero real number such that  $x^4 + \frac{1}{x^4}$  and  $x^5 + \frac{1}{x^5}$  are both rational numbers. Prove that  $x + \frac{1}{x}$  is a rational number.

**Solution.** For a natural number  $k$  let  $T_k = x^k + 1/x^k$ . Note that  $T_4 T_2 = T_2 + T_6$  and  $T_8 T_2 = T_{10} + T_6$ . Therefore  $T_2(T_8 - T_4 + 1) = T_{10}$ . Since  $T_{2k} = T_k^2 + 2$  it follows that  $T_8, T_{10}$  are rational numbers and hence  $T_2, T_6$  are also rational numbers. Since  $T_5 T_1 = T_4 + T_6$  it follows that  $T_1$  is a rational number.  $\square$

5. In a triangle  $ABC$ , let  $H$  denote its orthocentre. Let  $P$  be the reflection of  $A$  with respect to  $BC$ . The circumcircle of triangle  $ABP$  intersects the line  $BH$  again at  $Q$ , and the circumcircle of triangle  $ACP$  intersects the line  $CH$  again at  $R$ . Prove that  $H$  is the incentre of triangle  $PQR$ .

**Solution.** Since  $RACP$  is a cyclic quadrilateral it follows that  $\angle RPA = \angle RCA = 90^\circ - \angle A$ . Similarly, from cyclic quadrilateral  $BAQP$  we get  $\angle QPA = 90^\circ - \angle A$ . This shows that  $PH$  is the angular bisector of  $\angle RPQ$ .

We next show that  $R, A, Q$  are collinear. For this, note that  $\angle BPC = \angle A$ . Since  $\angle BHC = 180^\circ - \angle A$  it follows that  $BHCP$  is a cyclic quadrilateral. Therefore  $\angle RAP + \angle QAP = \angle RCP + \angle QBP = 180^\circ$ . This proves that  $R, A, Q$  are collinear.

Now  $\angle QRC = \angle ARC = \angle APC = \angle PAC = \angle PRC$ . This proves that  $RC$  is the angular bisector of  $\angle PRQ$  and hence  $H$  is the incenter of triangle  $PQR$ .  $\square$

6. Suppose that the vertices of a regular polygon of 20 sides are coloured with three colours – red, blue and green – such that there are exactly three red vertices. Prove that there are three vertices  $A, B, C$  of the polygon having the same colour such that triangle  $ABC$  is isosceles.

**Solution.** Since there are exactly three vertices, among the remaining 17 vertices there are nine of them of the same colour, say blue. We can divide the vertices of the regular 20-gon into four disjoint sets such that each set consists of vertices that form a regular pentagon. Since there are nine blue points, at least one of these sets will have three blue points. Since any three points on a pentagon form an isosceles triangle, the statement follows.  $\square$

# Regional Mathematical Olympiad-2014

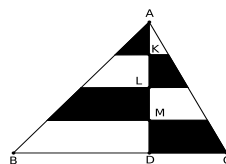
Time: 3 hours

December 07, 2014

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

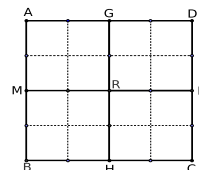
1. Let  $ABC$  be a triangle and let  $AD$  be the perpendicular from  $A$  on to  $BC$ . Let  $K, L, M$  be points on  $AD$  such that  $AK = KL = LM = MD$ . If the sum of the areas of the shaded regions is equal to the sum of the areas of the unshaded regions, prove that  $BD = DC$ .



2. Let  $a_1, a_2, \dots, a_{2n}$  be an arithmetic progression of positive real numbers with common difference  $d$ . Let  
 (i)  $a_1^2 + a_3^2 + \dots + a_{2n-1}^2 = x$ , (ii)  $a_2^2 + a_4^2 + \dots + a_{2n}^2 = y$ , and  
 (iii)  $a_n + a_{n+1} = z$ .  
 Express  $d$  in terms of  $x, y, z, n$ .

3. Suppose for some positive integers  $r$  and  $s$ , the digits of  $2^r$  is obtained by permuting the digits of  $2^s$  in decimal expansion. Prove that  $r = s$ .

4. Is it possible to write the numbers  $17, 18, 19, \dots, 32$  in a  $4 \times 4$  grid of unit squares, with one number in each square, such that the product of the numbers in each  $2 \times 2$  sub-grids  $AMRG$ ,  $GRND$ ,  $MBHR$  and  $RHCN$  is **divisible** by 16?



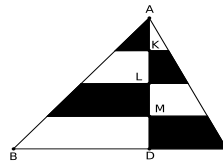
5. Let  $ABC$  be an acute-angled triangle and let  $H$  be its ortho-centre. For any point  $P$  on the circum-circle of triangle  $ABC$ , let  $Q$  be the point of intersection of the line  $BH$  with the line  $AP$ . Show that there is a unique point  $X$  on the circum-circle of  $ABC$  such that for every point  $P \neq A, B$ , the circum-circle of  $HQP$  pass through  $X$ .
6. Let  $x_1, x_2, \dots, x_{2014}$  be positive real numbers such that  $\sum_{j=1}^{2014} x_j = 1$ . Determine with proof the smallest constant  $K$  such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1.$$

—0—

## Solutions to RMO-2014 problems

1. Let  $ABC$  be a triangle and let  $AD$  be the perpendicular from  $A$  on to  $BC$ . Let  $K, L, M$  be points on  $AD$  such that  $AK = KL = LM = MD$ . If the sum of the areas of the shaded regions is equal to the sum of the areas of the unshaded regions, prove that  $BD = DC$ .



**Solution:** let  $BD = 4x$ ,  $DC = 4y$  and  $AD = 4h$ . Then the sum of the areas of the shaded regions is

$$\frac{1}{2}h(x + (y + 2y) + (2x + 3x) + (3y + 4y)) = \frac{h(6x + 10y)}{2}.$$

The sum of the areas of the unshaded regions is

$$\frac{1}{2}h(y + (x + 2x) + (2y + 3y) + (3x + 4x)) = \frac{h(10x + 6y)}{2}.$$

Therefore the given condition implies that

$$6x + 10y = 10x + 6y.$$

This gives  $x = y$ . Hence  $BD = DC$ .

2. Let  $a_1, a_2, \dots, a_{2n}$  be an arithmetic progression of positive real numbers with common difference  $d$ . Let

- (i)  $a_1^2 + a_3^2 + \dots + a_{2n-1}^2 = x$ , (ii)  $a_2^2 + a_4^2 + \dots + a_{2n}^2 = y$ , and  
(iii)  $a_n + a_{n+1} = z$ .

Express  $d$  in terms of  $x, y, z, n$ .

**Solution:** Observe that

$$y - x = (a_2^2 - a_1^2) + (a_4^2 - a_3^2) + \dots + (a_{2n}^2 - a_{2n-1}^2).$$

The general difference is

$$a_{2k}^2 - a_{2k-1}^2 = (a_{2k} + a_{2k-1})d = (2a_1 + ((2k-1) + (2k-2))d)d.$$

Therefore

$$y - x = (2na_1 + (1 + 2 + 3 + \dots + (2n-1))d)d = nd(2a_1 + (2n-1)d).$$

We also observe that

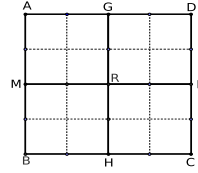
$$z = a_n + a_{n+1} = 2a_1 + (2n-1)d.$$

It follows that  $y - x = ndz$ . Hence  $d = (y - x)/nz$ .

3. Suppose for some positive integers  $r$  and  $s$ , the digits of  $2^r$  is obtained by permuting the digits of  $2^s$  in decimal expansion. Prove that  $r = s$ .

**Solution:** Suppose  $s \leq r$ . If  $s < r$  then  $2^s < 2^r$ . Since the number of digits in  $2^s$  and  $2^r$  are the same, we have  $2^r < 10 \times 2^s < 2^{s+4}$ . Thus we have  $2^s < 2^r < 2^{s+4}$  which gives  $r = s + 1$  or  $s + 2$  or  $s + 3$ . Since  $2^r$  is obtained from  $2^s$  by permuting its digits,  $2^r - 2^s$  is divisible by 9. If  $r = s + 1$ , we see that  $2^r - 2^s = 2^s$  and it is clearly not divisible by 9. Similarly,  $2^{s+2} - 2^s = 3 \times 2^s$  and  $2^{s+3} - 2^s = 7 \times 2^s$  and none of these is divisible by 9. We conclude that  $s < r$  is not possible. Hence  $r = s$ .

4. Is it possible to write the numbers  $17, 18, 19, \dots, 32$  in a  $4 \times 4$  grid of unit squares, with one number in each square, such that the product of the numbers in each  $2 \times 2$  sub-grids  $AMRG$ ,  $GRND$ ,  $MBHR$  and  $RHCN$  is **divisible** by 16?



**Solution:** NO! If the product in each  $2 \times 2$  sub-square is divisible by 16, then the product of all the numbers is divisible by  $16 \times 16 \times 16 \times 16 = 2^{16}$ . But it is easy to see that

$$17 \times 18 \times 19 \times \dots \times 32 = 2^{15}k,$$

where  $k$  is an odd number. Hence the product of all the numbers in the grid is not divisible by  $2^{16}$ .

5. Let  $ABC$  be an acute-angled triangle and let  $H$  be its ortho-centre. For any point  $P$  on the circum-circle of triangle  $ABC$ , let  $Q$  be the point of intersection of the line  $BH$  with the line  $AP$ . Show that there is a unique point  $X$  on the circum-circle of  $ABC$  such that for every point  $P \neq A, B$ , the circum-circle of  $HQP$  pass through  $X$ .

**Solution:** We consider two possibilities:  $Q$  lying between  $A$  and  $P$ ; and  $P$  lying between  $A$  and  $Q$ . (See the figures.)

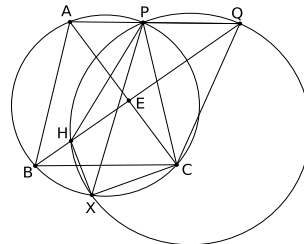
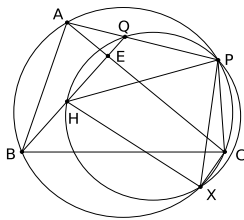
In the first case, we observe that

$$\angle HXC = \angle HXP + \angle PXC = \angle AQB + \angle PAC,$$

since  $Q, H, X, P$  are concyclic and  $P, A, X, C$  are also concyclic. Thus we get

$$\angle HXC = \angle AQE + \angle QAE = 90^\circ$$

because  $BE \perp AC$ .



In the second case, we have

$$\angle HXC = \angle HXP + \angle PXC = \angle HQP + \angle PAC;$$

the first follows from  $H, X, Q, P$  are concyclic; the second follows from the concyclicity of  $A, X, C, P$ . Again  $BE \perp AC$  shows that  $\angle HXC = 90^\circ$ .

Thus for any point  $P \neq A, B$  on the circumcircle of  $ABC$ , the point  $X$  of intersection of the circumcircles of  $ABC$  and  $HPQ$  is such that  $\angle HXC = 90^\circ$ . This means  $X$  is precisely the point of intersection of the circumcircles of  $HEC$  and  $ABC$ , which is independent of  $P$ .

6. Let  $x_1, x_2, \dots, x_{2014}$  be positive real numbers such that  $\sum_{j=1}^{2014} x_j = 1$ . Determine with proof the smallest constant  $K$  such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1.$$

**Solution:** Let us take the general case:  $\{x_1, x_2, \dots, x_n\}$  are positive real numbers such that  $\sum_{k=1}^n x_k = 1$ . Then

$$\sum_{k=1}^n \frac{x_k^2}{1 - x_k} = \sum_{k=1}^n \frac{x_k^2 - 1}{1 - x_k} + \sum_{k=1}^n \frac{1}{1 - x_k} = \sum_{k=1}^n (-1 - x_k) + \sum_{k=1}^n \frac{1}{1 - x_k}.$$

Now the first sum is  $-n - 1$ . We can estimate the second sum using AM-HM inequality:

$$\sum_{k=1}^n \frac{1}{1 - x_k} \geq \frac{n^2}{\sum_{k=1}^n (1 - x_k)} = \frac{n^2}{n - 1}.$$

Thus we obtain

$$\sum_{k=1}^n \frac{x_k^2}{1 - x_k} \geq -(1 + n) + \frac{n^2}{n - 1} = \frac{1}{n - 1}.$$

Here equality holds if and only if all  $x_j$ 's are equal. Thus we get the smallest constant  $K$  such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1$$

to be  $2014 - 1 = 2013$ .

—————0—————

# Regional Mathematical Olympiad-2014

Time: 3 hours

December 07, 2014

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. In an acute-angled triangle  $ABC$ ,  $\angle ABC$  is the largest angle. The perpendicular bisectors of  $BC$  and  $BA$  intersect  $AC$  at  $X$  and  $Y$  respectively. Prove that circumcentre of triangle  $ABC$  is incentre of triangle  $BXY$ .
2. Let  $x, y, z$  be positive real numbers. Prove that

$$\frac{y^2 + z^2}{x} + \frac{z^2 + x^2}{y} + \frac{x^2 + y^2}{z} \geq 2(x + y + z).$$

3. Find all pairs of  $(x, y)$  of positive integers such that  $2x + 7y$  divides  $7x + 2y$ .
4. For any positive integer  $n > 1$ , let  $P(n)$  denote the largest prime not exceeding  $n$ . Let  $N(n)$  denote the next prime larger than  $P(n)$ . (For example  $P(10) = 7$  and  $N(10) = 11$ , while  $P(11) = 11$  and  $N(11) = 13$ .) If  $n + 1$  is a prime number, prove that the value of the sum

$$\frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)} = \frac{n-1}{2n+2}.$$

5. Let  $ABC$  be a triangle with  $AB > AC$ . Let  $P$  be a point on the line  $AB$  beyond  $A$  such that  $AP + PC = AB$ . Let  $M$  be the mid-point of  $BC$  and let  $Q$  be the point on the side  $AB$  such that  $CQ \perp AM$ . Prove that  $BQ = 2AP$ .
6. Let  $n$  be an odd positive integer and suppose that each square of an  $n \times n$  grid is arbitrarily filled with either by 1 or by  $-1$ . Let  $r_j$  and  $c_k$  denote the product of all numbers in  $j$ -th row and  $k$ -th column respectively,  $1 \leq j, k \leq n$ . Prove that

$$\sum_{j=1}^n r_j + \sum_{k=1}^n c_k \neq 0.$$

## Solutions to RMO-2014 problems

1. In an acute-angled triangle  $ABC$ ,  $\angle ABC$  is the largest angle. The perpendicular bisectors of  $BC$  and  $BA$  intersect  $AC$  at  $X$  and  $Y$  respectively. Prove that circumcentre of triangle  $ABC$  is incentre of triangle  $BXY$ .

**Solution:** Let  $D$  and  $E$  be the mid-points of  $BC$  and  $AB$  respectively. Since  $X$  lies on the perpendicular bisector of  $BC$ , we have  $XB = XC$ . Since  $XD \perp BC$  and  $XB = XC$ , it follows that  $XD$  bisects  $\angle BXC$ . Similarly,  $YE$  bisects  $\angle BYA$ . Hence the point of intersection of  $XD$  and  $YE$  is the incentre of  $\triangle BXY$ . But this point of intersection is also the circumcentre of  $\triangle ABC$ , being the point of intersection of perpendicular bisectors of  $BC$  and  $AB$ .

2. Let  $x, y, z$  be positive real numbers. Prove that

$$\frac{y^2 + z^2}{x} + \frac{z^2 + x^2}{y} + \frac{x^2 + y^2}{z} \geq 2(x + y + z).$$

**Solution:** We write the inequality in the form

$$\frac{x^2}{y} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{y} + \frac{z^2}{x} + \frac{x^2}{z} \geq 2(x + y + z).$$

We observe that  $x^2 + y^2 \geq 2xy$ . Hence  $x^2 + y^2 - xy \geq xy$ . Multiplying both sides by  $(x + y)$ , we get

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) \geq (x + y)xy.$$

Thus

$$\frac{x^2}{y} + \frac{y^2}{x} \geq x + y.$$

Similarly, we obtain

$$\frac{y^2}{z} + \frac{z^2}{y} \geq y + z, \quad \frac{z^2}{x} + \frac{x^2}{z} \geq x + y.$$

Adding three inequalities, we get the required result.

3. Find all pairs of  $(x, y)$  of positive integers such that  $2x + 7y$  divides  $7x + 2y$ .

**Solution:** Let  $d = \gcd(x, y)$ . Then  $x = dx_1$  and  $y = dy_1$ . We observe that  $2x + 7y$  divides  $7x + 2y$  if and only if  $2x_1 + 7y_1$  divides  $7x_1 + 2y_1$ . This means  $2x_1 + 7y_1$  should divide  $49x_1 + 14y_1$ . But  $2x_1 + 7y_1$  divides  $4x_1 + 14y_1$ . Hence  $2x_1 + 7y_1$  divides  $45x_1$ . Similarly, we can show that  $2x_1 + 7y_1$  divides  $45y_1$ . Hence  $2x_1 + 7y_1$  divides  $\gcd(45x_1, 45y_1) = 45 \gcd(x_1, y_1) = 45$ . Hence

$$2x_1 + 7y_1 = 9, 15 \text{ or } 45.$$

If  $2x_1 + 7y_1 = 9$ , then  $x_1 = 1, y_1 = 1$ . Similarly,  $2x_1 + 7y_1 = 15$  gives  $x_1 = 4, y_1 = 1$ . If  $2x_1 + 7y_1 = 45$ , then we get

$$(x_1, y_1) = (19, 1), (12, 3), (5, 5).$$

Thus all solutions are of the form

$$(x, y) = (d, d), (4d, d), (19d, d), (12d, 3d), (5d, 5d).$$

4. For any positive integer  $n > 1$ , let  $P(n)$  denote the largest prime not exceeding  $n$ . Let  $N(n)$  denote the next prime larger than  $P(n)$ . (For example  $P(10) = 7$  and  $N(10) = 11$ , while  $P(11) = 11$  and  $N(11) = 13$ .) If  $n + 1$  is a prime number, prove that the value of the sum

$$\frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)} = \frac{n-1}{2n+2}.$$

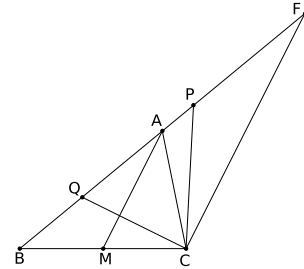
**Solution:** Let  $p$  and  $q$  be two consecutive primes,  $p < q$ . If we take any  $n$  such that  $p \leq n < q$ , we see that  $P(n) = p$  and  $N(n) = q$ . Hence the term  $\frac{1}{pq}$  occurs in the sum  $q - p$  times. The contribution from such terms is  $\frac{q-p}{pq} = \frac{1}{p} - \frac{1}{q}$ . Since  $n + 1$  is prime, we obtain

$$\begin{aligned} & \frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)} \\ &= \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{p} - \frac{1}{n+1} \right) = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2n+2}. \end{aligned}$$

Here  $p$  is used for the prime preceeding  $n + 1$ .

5. Let  $ABC$  be a triangle with  $AB > AC$ . Let  $P$  be a point on the line  $AB$  beyond  $A$  such that  $AP + PC = AB$ . Let  $M$  be the mid-point of  $BC$  and let  $Q$  be the point on the side  $AB$  such that  $CQ \perp AM$ . Prove that  $BQ = 2AP$ .

**Solution:** Extend  $BP$  to  $F$  such  $PF = PC$ . Then  $AF = AP + PF = AP + PC = AB$ . Hence  $A$  is the mid-point of  $BF$ . Since  $M$  is the mid-point of  $BC$ , it follows that  $AM \parallel FC$ . But  $AM \perp CQ$ . Hence  $FC \perp CQ$  at  $C$ . Therefore  $QCF$  is a right-angled triangle. Since  $PC = PF$ , it follows that  $\angle PCF = \angle PFC$ . Hence  $\angle PQC = \angle PCQ$  which gives  $PQ = PC = PF$ . This implies that  $P$  is the mid-point of  $QF$ .



Thus we have  $AP + AQ = PF$  and  $BQ + QA = AP + PF$ . This gives

$$2AP + AQ = PF + AP = BQ + QA.$$

We conclude that  $BQ = 2AP$ .

6. Suppose  $n$  is odd and each square of an  $n \times n$  grid is arbitrarily filled with either by 1 or by  $-1$ . Let  $r_j$  and  $c_k$  denote the product of all numbers in  $j$ -th row and  $k$ -th column respectively,  $1 \leq j, k \leq n$ . Prove that

$$\sum_{j=1}^n r_j + \sum_{k=1}^n c_k \neq 0.$$

**Solution:** Suppose we change  $+1$  to  $-1$  in a square. Then the product of the numbers in that row changes sign. Similarly, the product of numbers in the column also changes sign. Hence the sum

$$S = \sum_{j=1}^n r_j + \sum_{k=1}^n c_k$$

decreases by 4 or increases by 4 or remains same. Hence the new sum is congruent to the old sum modulo 4. Let us consider the situation when all the squares have  $+1$ . Then  $S = n + n = 2n = 2(2m + 1) = 4m + 2$ . This means the sum  $S$  is always of the form  $4l + 2$  for any configuration. Therefore the sum is not equal to 0.

————-00————-

# Regional Mathematical Olympiad-2014

Time: 3 hours

December 07, 2014

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be an acute-angled triangle and suppose  $\angle ABC$  is the largest angle of the triangle. Let  $R$  be its circumcentre. Suppose the circumcircle of triangle  $ARB$  cuts  $AC$  again in  $X$ . Prove that  $RX$  is perpendicular to  $BC$ .
2. Find all real numbers  $x$  and  $y$  such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y + 1).$$

3. Prove that there does not exist any positive integer  $n < 2310$  such that  $n(2310 - n)$  is a multiple of 2310.
4. Find all positive real numbers  $x, y, z$  such that

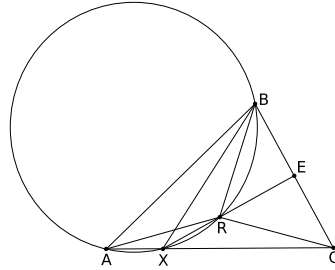
$$2x - 2y + \frac{1}{z} = \frac{1}{2014}, \quad 2y - 2z + \frac{1}{x} = \frac{1}{2014}, \quad 2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

5. Let  $ABC$  be a triangle. Let  $X$  be on the segment  $BC$  such that  $AB = AX$ . Let  $AX$  meet the circumcircle  $\Gamma$  of triangle  $ABC$  again at  $D$ . Show that the circumcentre of  $\triangle BDX$  lies on  $\Gamma$ .
6. For any natural number  $n$ , let  $S(n)$  denote the sum of the digits of  $n$ . Find the number of all 3-digit numbers  $n$  such that  $S(S(n)) = 2$ .

## Solutions to RMO-2014 problems

- Let  $ABC$  be an acute-angled triangle and suppose  $\angle ABC$  is the largest angle of the triangle. Let  $R$  be its circumcentre. Suppose the circumcircle of triangle  $ARB$  cuts  $AC$  again in  $X$ . Prove that  $RX$  is perpendicular to  $BC$ .

**Solution:** Extend  $RX$  to meet  $BC$  in  $E$ . We show that  $\angle XEC = 90^\circ$ . Join  $RA$ ,  $RB$  and  $BX$ . Observe that  $\angle AXB = \angle ARB = 2\angle C$  and  $\angle BXR = \angle BAR = 90^\circ - \angle C$ . Hence  $\angle EXC = 180^\circ - 2\angle C - (90^\circ - \angle C) = 90^\circ - \angle C$ . This shows that  $\angle CEX = 90^\circ$ .



- Find all real numbers  $x$  and  $y$  such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y + 1).$$

**Solution:** We write the inequality in the form

$$2x^2 + 4y^2 + 1 - 4xy - 2x \leq 0.$$

Thus  $(x^2 - 4xy + 4y^2) + (x^2 - 2x + 1) \leq 0$ . Hence

$$(x - 2y)^2 + (x - 1)^2 \leq 0.$$

Since  $x, y$  are real, we know that  $(x - 2y)^2 \geq 0$  and  $(x - 1)^2 \geq 0$ . Hence it follows that  $(x - 2y)^2 = 0$  and  $(x - 1)^2 = 0$ . Therefore  $x = 1$  and  $y = 1/2$ .

- Prove that there does not exist any positive integer  $n < 2310$  such that  $n(2310 - n)$  is a multiple of 2310.

**Solution:** Suppose there exists  $n$  such that  $0 < n < 2310$  and  $n(2310 - n) = 2310k$ . Then  $n^2 = 2310(n - k)$ . But  $2310 = 2 \times 3 \times 5 \times 7 \times 11$ , the product of primes. Hence  $n - k = 2310l^2$  for some  $l$ . But  $n < 2310$  and hence  $n - k < 2310$ . Hence  $l = 0$ . This forces  $n = k$  and hence  $n^2 = 2310(n - k) = 0$ . Thus  $n = 0$  and we have a contradiction.

- Find all positive real numbers  $x, y, z$  such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2014}, \quad 2y - 2z + \frac{1}{x} = \frac{1}{2014}, \quad 2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

**Solution:** Adding the three equations, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2014}.$$

We can also write the equations in the form

$$2zx - 2zy + 1 = \frac{z}{2014}, \quad 2xy - 2xz + 1 = \frac{x}{2014}, \quad 2yz - 2yx + 1 = \frac{y}{2014}.$$

Adding these, we also get

$$2014 \times 3 = x + y + z.$$

Therefore

$$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) = \frac{3}{2014} \times (2014 \times 3) = 9.$$

Using AM-GM inequality, we therefore obtain

$$9 = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) \geq 9 \times (xyz)^{1/3} \left(\frac{1}{xyz}\right)^{1/3} = 9.$$

Hence equality holds in AM-GM inequality and we conclude  $x = y = z$ . Thus

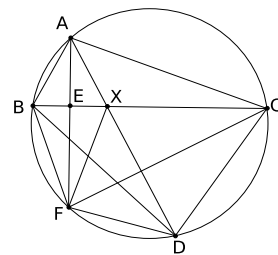
$$\frac{1}{x} = \frac{1}{2014}$$

which gives  $x = 2014$ . We conclude

$$x = 2014, \quad y = 2014, \quad z = 2014.$$

5. Let  $ABC$  be a triangle. Let  $X$  be on the segment  $BC$  such that  $AB = AX$ . Let  $AX$  meet the circumcircle  $\Gamma$  of triangle  $ABC$  again at  $D$ . Show that the circumcentre of  $\triangle BD X$  lies on  $\Gamma$ .

**Solution:** Draw perpendicular from  $A$  to  $BC$  and extend it to meet  $\Gamma$  in  $F$ . We show that  $F$  is the circumcentre of  $\triangle BD X$ . Since  $AB = AX$ , we observe that  $F$  lies on the perpendicular bisector of  $BX$ . Join  $CF$  and  $CD$ . We observe that  $\angle ABX = \angle CDX$  and  $\angle AXB = \angle CXD$ . Hence  $\triangle ABX$  is similar to  $\triangle CDX$ . In particular  $\triangle CDX$  is isosceles.



Moreover,  $\angle BCF = \angle BAF$  and  $\angle DCF = \angle DAF$ . Since  $AF$  is the perpendicular bisector of  $BX$ , it also bisects  $\angle BAX$ . It follows that  $CF$  bisects  $\angle DCX$  and hence  $F$  lies on the perpendicular bisector of  $DX$ . Together  $F$  is the circumcentre of  $\triangle B X D$ .

6. For any natural number  $n$ , let  $S(n)$  denote the sum of the digits of  $n$ . Find the number of all 3-digit numbers  $n$  such that  $S(S(n)) = 2$ .

**Solution:** Observe that  $S(S(n)) = 2$  implies that  $S(n) = 2, 11$  or  $20$ . Hence we have to find the number of all 3 digit numbers  $\overline{abc}$  such that  $a + b + c = 2, 11$

and 20. In fact we can enumerate all these:

$a + b + c = 2$ :  $\overline{abc} = 101, 110, 200$ ;  
 $a + b + c = 11$ :  $\overline{abc} = 902, 920, 290, 209, 911, 191, 119, 803, 830, 308, 380$ ,  
 $812, 821, 182, 128, 218, 281, 731, 713, 317, 371, 137, 173, 722, 272, 227, 740, 704$ ,  
 $407, 470, 650, 605, 560, 506, 641, 614, 416, 461, 164, 146, 623, 632, 362, 326, 263, 236$ ;  
 $a + b + c = 20$ :  $\overline{abc} = 992, 929, 299, 983, 938, 398, 389, 839, 893, 974, 947, 794, 749$ ,  
 $479, 497, 965, 956, 659, 695, 596, 569, 884, 848, 488$ ,  
 $875, 875, 785, 758, 578, 587, 866, 686, 668, 776, 767, 677$ .

There are totally 85 three digit numbers having second digital sum equal to 2.

————-00————-

# Regional Mathematical Olympiad-2014

Time: 3 hours

December 07, 2014

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABCD$  be an isosceles trapezium having an incircle; let  $AB$  and  $CD$  be the parallel sides and let  $CE$  be the perpendicular from  $C$  on to  $AB$ . Prove that  $CE$  is equal to the geometric mean of  $AB$  and  $CD$ .
2. If  $x$  and  $y$  are positive real numbers, prove that

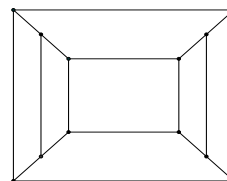
$$4x^4 + 4y^3 + 5x^2 + y + 1 \geq 12xy.$$

3. Determine all pairs  $m > n$  of positive integers such that

$$1 = \gcd(n+1, m+1) = \gcd(n+2, m+2) = \cdots = \gcd(m, 2m-n).$$

4. What is the minimal area of a right-angled triangle whose inradius is 1 unit?
5. Let  $ABC$  be an acute-angled triangle and let  $I$  be its incentre. Let the incircle of triangle  $ABC$  touch  $BC$  in  $D$ . The incircle of the triangle  $ABD$  touches  $AB$  in  $E$ ; the incircle of the triangle  $ACD$  touches  $BC$  in  $F$ . Prove that  $B, E, I, F$  are concyclic.

6. In the adjacent figure, can the numbers  $1, 2, 3, 4, \dots, 18$  be placed, one on each line segment, such that the sum of the numbers on the three line segments meeting at each point is divisible by 3?



## Solutions to RMO-2014 problems

- Let  $ABCD$  be an isosceles trapezium having an incircle; let  $AB$  and  $CD$  be the parallel sides and let  $CE$  be the perpendicular from  $C$  on to  $AB$ . Prove that  $CE$  is equal to the geometric mean of  $AB$  and  $CD$ .

**Solution:** Since  $ABCD$  has incircle, we have  $AB + CD = AD + BC$ . We also know that  $AD = BC$  and  $\angle A = \angle B$ . Draw  $DF \perp AB$ . Then  $\triangle DFC \cong \triangle CEB$ . Hence  $FE = CD$ , and  $AF = EB$ . Now

$$CE^2 = BC^2 - BE^2.$$

Observe  $2BC = BC + AD = AB + CD = 2FE + 2EB$ . Hence  $BC = FE + EB$ . Thus

$$CE^2 = (FE + EB)^2 - BE^2 = (FE + 2EB)FE = AB \cdot CD.$$

This shows that  $CE$  is the geometric mean of  $AB$  and  $CD$ .

- If  $x$  and  $y$  are positive real numbers, prove that

$$4x^4 + 4y^3 + 5x^2 + y + 1 \geq 12xy.$$

**Solution:** We have from AM-GM inequality,

$$4x^4 + 1 \geq 4x^2, \quad 4y^3 + y = y(4y^2 + 1) \geq 4y^2.$$

Hence

$$\begin{aligned} 4x^4 + 4y^3 + 5x^2 + y + 1 &\geq 4x^2 + 4y^2 + 5x^2 \\ &= 9x^2 + 4y^2 \\ &\geq 2(\sqrt{9 \times 4})xy \\ &= 12xy. \end{aligned}$$

- Determine all pairs  $m > n$  of positive integers such that

$$1 = \gcd(n+1, m+1) = \gcd(n+2, m+2) = \cdots = \gcd(m, 2m-n).$$

**Solution:** Observe that  $1 = \gcd(n+r, m+r) = \gcd(n+r, m-n)$ . Thus each of the  $m-n$  consecutive positive integers  $n+1, n+2, \dots, m$  is coprime to  $m-n$ . Since one of these is necessarily a multiple of  $m-n$ , this is possible only when  $m-n = 1$ . Hence each pair is of the form  $(n, n+1)$ , where  $n \in \mathbb{N}$ .

- What is the minimal area of a right-angled triangle whose inradius is 1 unit?

**Solution:** Let  $ABC$  be the right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be its incentre and  $D$  be the point where the incircle touches  $AB$ . Then  $s - b = AD = r = 1$ . We also know that  $[ABC] = rs = r(a + b + c)/2$  and  $[ABC] = ac/2$ . Thus

$$\frac{ac}{2} = \frac{a + b + c}{2} = (a + c) - \frac{a + c - b}{2} = (a + c) - 1.$$

Using AM-GM inequality, we get

$$\frac{ac}{2} = a + c - 1 \geq 2\sqrt{ac} - 1.$$

Taking  $\sqrt{ac} = x$ , we get  $x^2 - 4x + 2 \geq 0$ . Hence

$$x \geq \frac{4 + 2\sqrt{2}}{2} = 2 + \sqrt{2}.$$

Finally,

$$[ABC] = \frac{ac}{2} \geq \frac{(2 + \sqrt{2})^2}{2} = 3 + 2\sqrt{2}.$$

Thus the least area of such a triangle is  $3 + 2\sqrt{2}$ .

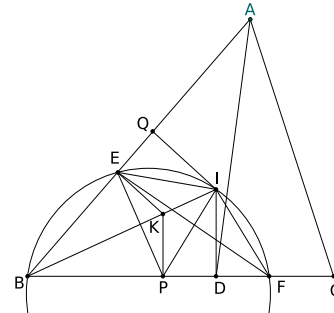
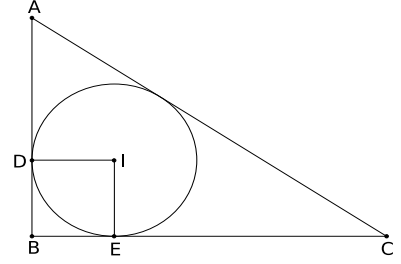
5. Let  $ABC$  be an acute-angled triangle and let  $I$  be its incentre. Let the incircle of triangle  $ABC$  touch  $BC$  in  $D$ . The incircle of the triangle  $ABD$  touches  $AB$  in  $E$ ; the incircle of the triangle  $ACD$  touches  $BC$  in  $F$ . Prove that  $B, E, I, F$  are concyclic.

**Solution:** We know  $BD = s - b$  and  $DC = s - c$ , where  $s$  is the semiperimeter of  $\triangle ABC$ . Let the incircle of  $\triangle ABD$  touch  $BC$  in  $P$  and let  $AD = l$ . Then

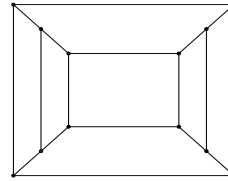
$$DP = \frac{l + BD - c}{2} = \frac{l + s - c - b}{2}.$$

Similarly, we can compute  $DF = \frac{l + s - c - b}{2}$ . Therefore  $DP = DF$ . But  $ID \perp BC$ . Hence  $I$  is on the perpendicular bisector of  $PF$ . This gives  $IP = IF$ .

Draw  $IQ \perp AB$ . Then  $B, Q, I, D$  are concyclic so that  $\angle QID = 180^\circ - \angle B$ . Since  $DP = DF$  and  $IP = IF$ , the triangles  $IDP$  and  $IDF$  are congruent. But  $IDP$  is congruent to  $IQE$ . It follows that  $\triangle IDF \cong \triangle IQE$ . This shows that  $\angle QIE = \angle DIF$ . Therefore  $\angle QID = \angle EIF$ . But  $\angle QID = 180^\circ - \angle B$ . Hence  $\angle EIF = 180^\circ - \angle B$ . Therefore  $B, E, I, F$  are concyclic.



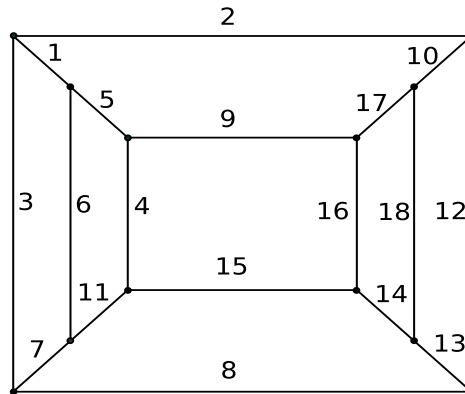
6. In the adjacent figure, can the numbers  $1, 2, 3, 4, \dots, 18$  be placed, one on each line segment, such that the sum of the numbers on the three line segments meeting at each point is divisible by 3?



**Solution:** We group the numbers 1 to 18 in to 3 groups: those leaving remainder 0 when divided by 3; those leaving remainder 1; and those leaving remainder 2. Thus the groups are:

$$\{3, 6, 9, 12, 15, 18\}, \{1, 4, 7, 10, 13, 16\}, \{2, 5, 8, 11, 14, 17\}$$

Now we put the numbers in such a way that each of the three line segments converging to a vertex gets one number from each set. For example, here is one such arrangement:



—00—

## Regional Mathematical Olympiad 2014 (Mumbai region)

- There are six questions in this question paper. Answer all questions.
- Each question carries 10 points.
- Use of protractors, calculators, mobile phone is forbidden.
- Time allotted: 3 hours

1. Three positive real numbers  $a, b, c$  are such that  $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$ . Can  $a, b, c$  be the lengths of the sides of a triangle? Justify your answer.
2. The roots of the equation

$$x^3 - 3ax^2 + bx + 18c = 0$$

form a non-constant arithmetic progression and the roots of the equation

$$x^3 + bx^2 + x - c^3 = 0$$

form a non-constant geometric progression. Given that  $a, b, c$  are real numbers, find all positive integral values  $a$  and  $b$ .

3. Let  $ABC$  be an acute-angled triangle in which  $\angle ABC$  is the largest angle. Let  $O$  be its circumcentre. The perpendicular bisectors of  $BC$  and  $AB$  meet  $AC$  at  $X$  and  $Y$  respectively. The internal bisectors of  $\angle AXB$  and  $\angle BYC$  meet  $AB$  and  $BC$  at  $D$  and  $E$  respectively. Prove that  $BO$  is perpendicular to  $AC$  if  $DE$  is parallel to  $AC$ .
4. A person moves in the  $x - y$  plane moving along points with integer co-ordinates  $x$  and  $y$  only. When she is at point  $(x, y)$ , she takes a step based on the following rules:
  - (a) if  $x + y$  is even she moves to either  $(x + 1, y)$  or  $(x + 1, y + 1)$ ;
  - (b) if  $x + y$  is odd she moves to either  $(x, y + 1)$  or  $(x + 1, y + 1)$ .

How many distinct paths can she take to go from  $(0, 0)$  to  $(8, 8)$  given that she took exactly three steps to the right  $((x, y) \text{ to } (x + 1, y))$ ?

5. Let  $a, b, c$  be positive numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1.$$

Prove that  $(1 + a^2)(1 + b^2)(1 + c^2) \geq 125$ . When does the equality hold?

6. Let  $D, E, F$  be the points of contact of the incircle of an acute-angled triangle  $ABC$  with  $BC, CA, AB$  respectively. Let  $I_1, I_2, I_3$  be the incentres of the triangles  $AFE, BDF, CED$ , respectively. Prove that the lines  $I_1D, I_2E, I_3F$  are concurrent.

**GOOD LUCK**

### Solutions to problems of RMO 2014 (Mumbai region)

1. **Three positive real numbers  $a, b, c$  are such that  $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$ . Can  $a, b, c$  be the lengths of the sides of a triangle? Justify your answer.**

#### Solution

No. Note that  $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = (a - 2b)^2 + (b - 2c)^2 = 0 \Rightarrow a : b : c = 4 : 2 : 1 \Rightarrow b + c : a = 3 : 4$ . The triangle inequality is violated.

2. **The roots of the equation**

$$x^3 - 3ax^2 + bx + 18c = 0$$

**form a non-constant arithmetic progression and the roots of the equation**

$$x^3 + bx^2 + x - c^3 = 0$$

**form a non-constant geometric progression. Given that  $a, b, c$  are real numbers, find all positive integral values  $a$  and  $b$ .**

#### Solution

Let  $\alpha - d, \alpha, \alpha + d$  ( $d \neq 0$ ) be the roots of the first equation and let  $\beta/r, \beta, \beta r$  ( $r > 0$  and  $r \neq 1$ ) be the roots of the second equation. It follows that  $\alpha = a, \beta = c$  and

$$a^3 - ad^2 = -18c; \quad 3a^2 - d^2 = b, \quad (1)$$

$$c(1/r + 1 + r) = -b; \quad c^2(1/r + 1 + r) = 1. \quad (2)$$

Eliminating  $d, r$  and  $c$  yields

$$ab^2 - 2a^3b - 18 = 0, \quad (3)$$

whence  $b = a^2 \pm (1/a)\sqrt{a^6 + 18a}$ . For positive integral values of  $a$  and  $b$  it must be that  $a^6 + 18a$  is a perfect square. Let  $x^2 = a^6 + 18a$ . Then  $a^3 < x^2 < a^3 + 1$  for  $a > 2$  and hence no solution. For  $a = 1$  there is no solution. For  $a = 2, x = 10$  and  $b = 9$ . Thus the admissible pair is  $(a, b) = (2, 9)$ .

3. **Let  $ABC$  be an acute-angled triangle in which  $\angle ABC$  is the largest angle. Let  $O$  be its circumcentre. The perpendicular bisectors of  $BC$  and  $AB$  meet  $AC$  at  $X$  and  $Y$  respectively. The internal bisectors of  $\angle AXB$  and  $\angle BYC$  meet  $AB$  and  $BC$  at  $D$  and  $E$  respectively. Prove that  $BO$  is perpendicular to  $AC$  if  $DE$  is parallel to  $AC$ .**

#### Solution

Observe that triangles  $AYB$  and  $BXC$  are isosceles ( $AY = BY$  and  $BX = CX$ ). This implies  $\angle BYC = 2\angle BAC$  and  $\angle AXB = 2\angle ACB$ . Since  $XD$  and  $YE$  are angle bisectors we have  $\angle AXD = \angle ACB$  and  $\angle CYE = \angle CAB$ . Hence  $XD$  is parallel to  $BC$  and  $YE$  is parallel to  $AB$ . Therefore

$$\frac{CE}{EB} = \frac{CY}{AY} \quad (4)$$

and

$$\frac{AD}{DB} = \frac{AX}{CX}. \quad (5)$$

Now, if  $DE$  is parallel to  $AC$  then  $\frac{CE}{EB} = \frac{AD}{DB}$ . Therefore we must have

$$\frac{CY}{AY} = \frac{AX}{CX}. \quad (6)$$

But then

$$\frac{CY}{AY} + 1 = \frac{AX}{CX} + 1 \Rightarrow \frac{AC}{AY} = \frac{AC}{CX} \Rightarrow AY = CX. \quad (7)$$

Hence  $BY = AY = CX = BX$ . Thus  $\angle BXY = \angle BYX$  i.e  $\angle AXB = \angle BYC$  or  $\angle ACB = \angle BAC$  i.e triangle  $ABC$  is isosceles with  $AB = CB$ . Hence  $BO$  is the perpendicular bisector of  $AC$ .

4. **A person moves in the  $x - y$  plane moving along points with integer co-ordinates  $x$  and  $y$  only. When she is at point  $(x, y)$ , she takes a step based on the following rules:**

(a) **if  $x + y$  is even she moves to either  $(x + 1, y)$  or  $(x + 1, y + 1)$ ;**

(b) **if  $x + y$  is odd she moves to either  $(x, y + 1)$  or  $(x + 1, y + 1)$ .**

**How many distinct paths can she take to go from  $(0, 0)$  to  $(8, 8)$  given that she took exactly three steps to the right  $((x, y) \text{ to } (x + 1, y))$ ?**

### Solution

We note that she must also take three up steps and five diagonal steps. Now, a step to the right or an upstep changes the parity of the co-ordinate sum, and a diagonal step does not change it. Therefore, between two right steps there must be an upstep and similarly between two upsteps there must be a right step. We may, therefore write

$$HVHVHV$$

The diagonal steps may be distributed in any fashion before, in between and after the HV sequence. The required number is nothing but the number of ways of distributing 5 identical objects into 7 distinct boxes and is equal to  $\binom{11}{6}$ .

5. **Let  $a, b, c$  be positive numbers such that**

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1.$$

**Prove that  $(1 + a^2)(1 + b^2)(1 + c^2) \geq 125$ . When does the equality hold?**

### Solution

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1 \Rightarrow \frac{a}{1+a} \geq \frac{1}{1+b} + \frac{1}{1+c}. \quad (8)$$

Similarly,

$$\frac{b}{1+b} \geq \frac{1}{1+a} + \frac{1}{1+c}, \quad \frac{c}{1+c} \geq \frac{1}{1+a} + \frac{1}{1+b}. \quad (9)$$

Apply AM-GM to get that

$$\frac{a}{1+a} \geq \frac{2}{\sqrt{(1+b)(1+c)}}, \quad \frac{b}{1+b} \geq \frac{2}{\sqrt{(1+a)(1+c)}}, \quad \frac{c}{1+c} \geq \frac{2}{\sqrt{(1+a)(1+b)}}. \quad (10)$$

Multiplying these results we get

$$abc \geq 8. \quad (11)$$

Now take

$$F = (1 + a^2)(1 + b^2)(1 + c^2) \geq 1 + a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 + a^2b^2c^2 \quad (12)$$

and apply AM-GM to  $a^2, b^2, c^2$  and to  $a^2b^2, b^2c^2, c^2a^2$  to get

$$F \geq 1 + 3(a^2b^2c^2)^{1/3} + 3(a^4b^4c^4)^{1/3} + a^2b^2c^2 = [1 + (a^2b^2c^2)^{1/3}]^3 \geq [1 + 8^{2/3}]^3 = 125. \quad (13)$$

Wherein the equality holds when  $a = b = c = 2$ .

6. **Let  $D, E, F$  be the points of contact of the incircle of an acute-angled triangle  $ABC$  with  $BC, CA, AB$  respectively. Let  $I_1, I_2, I_3$  be the incentres of the triangles  $AFE, BDF, CED$ , respectively. Prove that the lines  $I_1D, I_2E, I_3F$  are concurrent.**

### Solution

Observe that  $\angle AFE = \angle AEF = 90^\circ - A/2$  and  $\angle FDE = \angle AEF = 90^\circ - A/2$ . Again  $\angle EI_1F = 90^\circ + A/2$ . Thus

$$\angle EI_1F + \angle FDE = 180^\circ.$$

Hence  $I_1$  lies on the incircle. Also

$$\angle I_1FE = (1/2)\angle AFE = (1/2)\angle AEF = \angle I_1EF. \quad (14)$$

Thus  $I_1E = I_1F$ . But then they are equal chords of a circle and so they must subtend equal angles at the circumference. Therefore  $\angle I_1DF = \angle I_1DE$  and so  $I_1D$  is the internal bisector of  $\angle FDE$ . Similarly we can show that  $I_2E$  and  $I_3F$  are internal bisectors of  $\angle DEF$  and  $\angle DFE$  respectively. Thus the three lines  $I_1D, I_2E, I_3F$  are concurrent at the incentre of triangle  $DEF$ .

# Regional Mathematical Olympiad-2015

Time: 3 hours

December 06, 2015

Instructions:

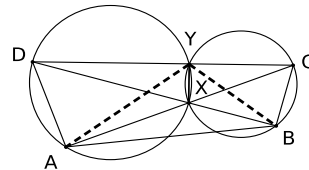
- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. In a cyclic quadrilateral  $ABCD$ , let the diagonals  $AC$  and  $BD$  intersect at  $X$ . Let the circumcircles of triangles  $AXD$  and  $BXC$  intersect again at  $Y$ . If  $X$  is the incentre of triangle  $ABY$ , show that  $\angle CAD = 90^\circ$ .
2. Let  $P_1(x) = x^2 + a_1x + b_1$  and  $P_2(x) = x^2 + a_2x + b_2$  be two quadratic polynomials with integer coefficients. Suppose  $a_1 \neq a_2$  and there exist integers  $m \neq n$  such that  $P_1(m) = P_2(n)$ ,  $P_2(m) = P_1(n)$ . Prove that  $a_1 - a_2$  is even.
3. Find all fractions which can be written simultaneously in the forms  $\frac{7k-5}{5k-3}$  and  $\frac{6l-1}{4l-3}$ , for some integers  $k, l$ .
4. Suppose 28 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?
5. Let  $ABC$  be a right triangle with  $\angle B = 90^\circ$ . Let  $E$  and  $F$  be respectively the mid-points of  $AB$  and  $AC$ . Suppose the incentre  $I$  of triangle  $ABC$  lies on the circumcircle of triangle  $AEF$ . Find the ratio  $BC/AB$ .
6. Find all real numbers  $a$  such that  $3 < a < 4$  and  $a(a - 3\{a\})$  is an integer. (Here  $\{a\}$  denotes the fractional part of  $a$ . For example  $\{1.5\} = 0.5$ ;  $\{-3.4\} = 0.6$ .)

## CRMO-2015 questions and solutions

1. In a cyclic quadrilateral  $ABCD$ , let the diagonals  $AC$  and  $BD$  intersect at  $X$ . Let the circumcircles of triangles  $AXD$  and  $BXC$  intersect again at  $Y$ . If  $X$  is the incentre of triangle  $ABY$ , show that  $\angle CAD = 90^\circ$ .

**Solution:** Given that  $X$  is the incentre of triangle  $ABY$ , we have  $\angle BAX = \angle XAY$ . Therefore,  $\angle BDC = \angle BAC = \angle BAX = \angle XAY = \angle XDY = \angle BDY$ . This shows that  $C, D, Y$  are collinear. Therefore,  $\angle CYX + \angle XYD = 180^\circ$ . But the left-hand side equals  $(180^\circ - \angle CBD) + (180^\circ - \angle CAD)$ . Since  $\angle CBD = \angle CAD$ , we obtain  $180^\circ = 360^\circ - 2\angle CAD$ . This shows that  $\angle CAD = 90^\circ$ .



2. Let  $P_1(x) = x^2 + a_1x + b_1$  and  $P_2(x) = x^2 + a_2x + b_2$  be two quadratic polynomials with integer coefficients. Suppose  $a_1 \neq a_2$  and there exist integers  $m \neq n$  such that  $P_1(m) = P_2(n)$ ,  $P_2(m) = P_1(n)$ . Prove that  $a_1 - a_2$  is even.

**Solution:** We have

$$\begin{aligned} m^2 + a_1m + b_1 &= n^2 + a_2n + b_2 \\ n^2 + a_1n + b_1 &= m^2 + a_2m + b_2. \end{aligned}$$

Hence

$$(a_1 - a_2)(m + n) = 2(b_2 - b_1), \quad (a_1 + a_2)(m - n) = 2(n^2 - m^2).$$

This shows that  $a_1 + a_2 = -2(n + m)$ . Hence

$$4(b_2 - b_1) = a_2^2 - a_1^2.$$

Since  $a_1 + a_2$  and  $a_1 - a_2$  have same parity, it follows that  $a_1 - a_2$  is even.

3. Find all fractions which can be written simultaneously in the forms  $\frac{7k-5}{5k-3}$  and  $\frac{6l-1}{4l-3}$ , for some integers  $k, l$ .

**Solution:** If a fraction is simultaneously in the forms  $\frac{7k-5}{5k-3}$  and  $\frac{6l-1}{4l-3}$ , we must have

$$\frac{7k-5}{5k-3} = \frac{6l-1}{4l-3}.$$

This simplifies to  $kl + 8k + l - 6 = 0$ . We can write this in the form

$$(k+1)(l+8) = 14.$$

Now 14 can be factored in 8 ways:  $1 \times 14$ ,  $2 \times 7$ ,  $7 \times 2$ ,  $14 \times 1$ ,  $(-1) \times (-14)$ ,  $(-2) \times (-7)$ ,  $(-7) \times (-2)$  and  $(-14) \times (-1)$ . Thus we get 8 pairs:

$$(k, l) = (13, -7), (6, -6), (1, -1), (0, 6), (-15, -9), (-8, -10), (-3, -15), (-2, -22).$$

These lead respectively to 8 fractions:

$$\frac{43}{31}, \quad \frac{31}{27}, \quad 1, \quad \frac{55}{39}, \quad \frac{5}{3}, \quad \frac{61}{43}, \quad \frac{19}{13}, \quad \frac{13}{9}.$$

4. Suppose 28 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

**Solution:** One can choose 3 objects out of 28 objects in  $\binom{28}{3}$  ways. Among these choices all would be together in 28 cases; exactly two will be together in  $28 \times 24$  cases. Thus three objects can be chosen such that no two adjacent in  $\binom{28}{3} - 28 - (28 \times 24)$  ways. Among these, further, two objects will be diametrically opposite in 14 ways and the third would be on either semicircle in a non adjacent portion in  $28 - 6 = 22$  ways. Thus required number is

$$\binom{28}{3} - 28 - (28 \times 24) - (14 \times 22) = 2268.$$

5. Let  $ABC$  be a right triangle with  $\angle B = 90^\circ$ . Let  $E$  and  $F$  be respectively the mid-points of  $AB$  and  $AC$ . Suppose the incentre  $I$  of triangle  $ABC$  lies on the circumcircle of triangle  $AEF$ . Find the ratio  $BC/AB$ .

**Solution:** Draw  $ID \perp AC$ . Then  $ID = r$ , the inradius of  $\triangle ABC$ . Observe  $EF \parallel BC$  and hence  $\angle AEF = \angle ABC = 90^\circ$ . Hence  $\angle AIF = 90^\circ$ . Therefore  $ID^2 = FD \cdot DA$ . If  $a > c$ , then  $FA > DA$  and we have

$$DA = s - a, \quad \text{and} \quad FD = FA - DA = \frac{b}{2} - (s - a).$$

Thus we obtain

$$r^2 = \frac{(b + c - a)(a - c)}{4}.$$

But  $r = (c + a - b)/2$ . Thus we obtain

$$(c + a - b)^2 = (b + c - a)(a - c).$$

Simplification gives  $3b = 3a + c$ . Squaring both sides and using  $b^2 = c^2 + a^2$ , we obtain  $4c = 3a$ . Hence  $BC/BA = a/c = 4/3$ .

(If  $a \leq c$ , then  $I$  lies outside the circumcircle of  $AEF$ .)

6. Find all real numbers  $a$  such that  $3 < a < 4$  and  $a(a - 3\{a\})$  is an integer. (Here  $\{a\}$  denotes the fractional part of  $a$ . For example  $\{1.5\} = 0.5$ ;  $\{-3.4\} = 0.6$ .)

**Solution:** Let  $a = 3 + f$ , where  $0 < f < 1$ . We are given that  $(3 + f)(3 - 2f)$  is an integer. This implies that  $2f^2 + 3f$  is an integer. Since  $0 < f < 1$ , we have  $0 < 2f^2 + 3f < 5$ . Therefore  $2f^2 + 3f$  can take 1, 2, 3 or 4. Equating  $2f^2 + 3f$  to each one of them and using  $f > 0$ , we get

$$f = \frac{-3 + \sqrt{17}}{4}, \quad \frac{1}{2}, \quad \frac{-3 + \sqrt{33}}{4}, \quad \frac{-3 + \sqrt{41}}{4}.$$

Therefore  $a$  takes the values:

$$a = 3 + \frac{-3 + \sqrt{17}}{4}, \quad 3\frac{1}{2}, \quad 3 + \frac{-3 + \sqrt{33}}{4}, \quad 3 + \frac{-3 + \sqrt{41}}{4}.$$

—————0—————

# Regional Mathematical Olympiad-2015

Time: 3 hours

December 06, 2015

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be a triangle. Let  $B'$  and  $C'$  denote respectively the reflection of  $B$  and  $C$  in the internal angle bisector of  $\angle A$ . Show that the triangles  $ABC$  and  $AB'C'$  have the same incentre.
2. Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial with real coefficients. Suppose there are real numbers  $s \neq t$  such that  $P(s) = t$  and  $P(t) = s$ . Prove that  $b - st$  is a root of the equation  $x^2 + ax + b - st = 0$ .
3. Find all integers  $a, b, c$  such that

$$a^2 = bc + 1, \quad b^2 = ca + 1.$$

4. Suppose 32 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?
5. Two circles  $\Gamma$  and  $\Sigma$  in the plane intersect at two distinct points  $A$  and  $B$ , and the centre of  $\Sigma$  lies on  $\Gamma$ . Let points  $C$  and  $D$  be on  $\Gamma$  and  $\Sigma$ , respectively, such that  $C, B$  and  $D$  are collinear. Let point  $E$  on  $\Sigma$  be such that  $DE$  is parallel to  $AC$ . Show that  $AE = AB$ .
6. Find all real numbers  $a$  such that  $4 < a < 5$  and  $a(a - 3\{a\})$  is an integer. (Here  $\{a\}$  denotes the fractional part of  $a$ . For example  $\{1.5\} = 0.5$ ;  $\{-3.4\} = 0.6$ .)

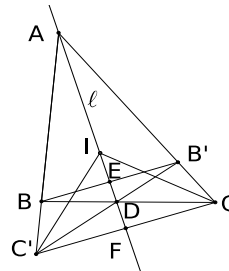
## CRMO-2015 questions and solutions

- Let  $ABC$  be a triangle. Let  $B'$  and  $C'$  denote respectively the reflection of  $B$  and  $C$  in the internal angle bisector of  $\angle A$ . Show that the triangles  $ABC$  and  $AB'C'$  have the same incentre.

**Solution:** Join  $BB'$  and  $CC'$ . Let the internal angle bisector  $\ell$  of  $\angle A$  meet  $BB'$  in  $E$  and  $CC'$  in  $F$ . Since  $B'$  is the reflection of  $B$  in  $\ell$ , we observe that  $BB' \perp \ell$  and  $BE = EB'$ . Hence  $B'$  lies on  $AC$ . Similarly,  $C'$  lies on the line  $AB$ .

Let  $D$  be the point of intersection of  $BC$  and  $B'C'$ . Observe that  $BB' \parallel C'C$ . Moreover the triangles  $ABC$  is congruent to  $AB'C'$ : this follows from the observation that  $AB = AB'$  and  $AC = AC'$  and the included angle  $\angle A$  is common. Hence  $BC' = B'C$  so that  $C'CB'B$  is an isosceles trapezium. This means that the intersection point  $D$  of its diagonal lies on the perpendicular bisector of its parallel sides. Thus  $\ell$  passes through  $D$ . We also observe that  $CD = C'D$ .

Let  $I$  be the incentre of  $\triangle ABC$ . This means that  $CI$  bisects  $\angle C$ . Hence  $AI/ID = AC/CD$ . But  $AC = AC'$  and  $CD = C'D$ . Hence we also get that  $AI/ID = AC'/C'D$ . This implies that  $C'I$  bisects  $\angle AC'B'$ . Therefore the two angle bisectors of  $\triangle AC'B'$  meet at  $I$ . This shows that  $I$  is also the incentre of  $\triangle AC'B'$ .



- Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial with real coefficients. Suppose there are real numbers  $s \neq t$  such that  $P(s) = t$  and  $P(t) = s$ . Prove that  $b - st$  is a root of the equation  $x^2 + ax + b - st = 0$ .

**Solution:** We have

$$\begin{aligned} s^2 + as + b &= t, \\ t^2 + at + b &= s. \end{aligned}$$

This gives

$$(s^2 - t^2) + a(s - t) = (t - s).$$

Since  $s \neq t$ , we obtain  $s + t + a = -1$ . Adding the equations, we obtain

$$s^2 + t^2 + a(s + t) + 2b = (s + t).$$

Therefore

$$(s + t)^2 - 2st + a(s + t) + 2b = (s + t).$$

Using  $s + t = -(1 + a)$ , we obtain

$$(1 + a)^2 - 2st - a(1 + a) + 2b = -1 - a.$$

Simplification gives  $st = 1 + a + b = P(1)$ . This shows that  $x = 1$  is a root of  $x^2 + ax + b - st = 0$ . Since the product of roots is  $b - st$ , the other root is  $b - st$ .

3. Find all integers  $a, b, c$  such that

$$a^2 = bc + 1, \quad b^2 = ca + 1.$$

**Solution:** Suppose  $a = b$ . Then we get one equation:  $a^2 = ac + 1$ . This reduces to  $a(a - c) = 1$ . Therefore  $a = 1, a - c = 1$ ; and  $a = -1, a - c = -1$ . Thus we get  $(a, b, c) = (1, 1, 0)$  and  $(-1, -1, 0)$ .

If  $a \neq b$ , subtracting the second relation from the first we get

$$a^2 - b^2 = c(b - a).$$

This gives  $a + b = -c$ . Substituting this in the first equation, we get

$$a^2 = b(-a - b) + 1.$$

Thus  $a^2 + b^2 + ab = 1$ . Multiplication by 2 gives

$$(a + b)^2 + a^2 + b^2 = 2.$$

Thus  $(a, b) = (1, -1), (-1, 1), (1, 0), (-1, 0), (0, 1), (0, -1)$ . We get respectively  $c = 0, 0, -1, 1, -1, 1$ . Thus we get the triples:

$$(a, b, c) = (1, 1, 0), (-1, -1, 0), (1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1).$$

4. Suppose 32 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

**Solution:** One can choose 3 objects out of 32 objects in  $\binom{32}{3}$  ways. Among these choices all would be together in 32 cases; exactly two will be together in  $32 \times 28$  cases. Thus three objects can be chosen such that no two adjacent in  $\binom{32}{3} - 32 - (32 \times 28)$  ways. Among these, further, two objects will be diametrically opposite in 16 ways and the third would be on either semicircle in a non adjacent portion in  $32 - 6 = 26$  ways. Thus required number is

$$\binom{32}{3} - 32 - (32 \times 28) - (16 \times 26) = 3616.$$

5. Two circles  $\Gamma$  and  $\Sigma$  in the plane intersect at two distinct points  $A$  and  $B$ , and the centre of  $\Sigma$  lies on  $\Gamma$ . Let points  $C$  and  $D$  be on  $\Gamma$  and  $\Sigma$ , respectively, such that  $C, B$  and  $D$  are collinear. Let point  $E$  on  $\Sigma$  be such that  $DE$  is parallel to  $AC$ . Show that  $AE = AB$ .

**Solution:** If  $O$  is the centre of  $\Sigma$ , then we have

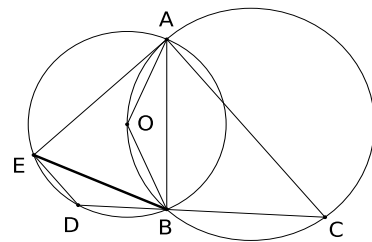
$$\begin{aligned} \angle AEB &= \frac{1}{2} \angle AOB = \frac{1}{2} (180^\circ - \angle ACB) \\ &= \frac{1}{2} \angle EDB = \frac{1}{2} (180^\circ - \angle EAB) = 90^\circ - \frac{1}{2} \angle EAB. \end{aligned}$$

But we know that  $\angle AEB + \angle EAB + \angle EBA = 180^\circ$ .

Therefore

$$\angle EBA = 180^\circ - \angle AEB - \angle EAB = 180^\circ - 90^\circ + \frac{1}{2} \angle EAB - \angle EAB = 90^\circ - \frac{1}{2} \angle EAB.$$

This shows that  $\angle AEB = \angle EBA$  and hence  $AE = AB$ .



6. Find all real numbers  $a$  such that  $4 < a < 5$  and  $a(a - 3\{a\})$  is an integer. (Here  $\{a\}$  denotes the fractional part of  $a$ . For example  $\{1.5\} = 0.5$ ;  $\{-3.4\} = 0.6$ .)

**Solution:** Let  $a = 4 + f$ , where  $0 < f < 1$ . We are given that  $(4 + f)(4 - 2f)$  is an integer. This implies that  $2f^2 + 4f$  is an integer. Since  $0 < f < 1$ , we have  $0 < 2f^2 + 4f < 6$ . Therefore  $2f^2 + 4f$  can take 1, 2, 3, 4 or 5. Equating  $2f^2 + 4f$  to each one of them and using  $f > 0$ , we get

$$f = \frac{-2 + \sqrt{6}}{2}, \frac{-2 + \sqrt{8}}{2}, \frac{-2 + \sqrt{10}}{2}, \frac{-2 + \sqrt{12}}{2}, \frac{-2 + \sqrt{14}}{2}.$$

Therefore  $a$  takes the values:

$$a = 3 + \frac{\sqrt{6}}{2}, 3 + \frac{\sqrt{8}}{2}, 3 + \frac{\sqrt{10}}{2}, 3 + \frac{\sqrt{12}}{2}, 3 + \frac{\sqrt{14}}{2}.$$

————-00————-

# Regional Mathematical Olympiad-2015

Time: 3 hours

December 06, 2015

## Instructions:

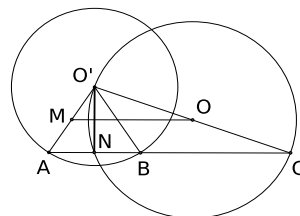
- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Two circles  $\Gamma$  and  $\Sigma$ , with centres  $O$  and  $O'$ , respectively, are such that  $O'$  lies on  $\Gamma$ . Let  $A$  be a point on  $\Sigma$  and  $M$  the midpoint of the segment  $AO'$ . If  $B$  is a point on  $\Sigma$  different from  $A$  such that  $AB$  is parallel to  $OM$ , show that the midpoint of  $AB$  lies on  $\Gamma$ .
2. Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial where  $a$  and  $b$  are real numbers. Suppose  $\langle P(-1)^2, P(0)^2, P(1)^2 \rangle$  is an arithmetic progression of integers. Prove that  $a$  and  $b$  are integers.
3. Show that there are infinitely many triples  $(x, y, z)$  of integers such that  $x^3 + y^4 = z^{31}$ .
4. Suppose 36 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?
5. Let  $ABC$  be a triangle with circumcircle  $\Gamma$  and incentre  $I$ . Let the internal angle bisectors of  $\angle A$ ,  $\angle B$  and  $\angle C$  meet  $\Gamma$  in  $A'$ ,  $B'$  and  $C'$  respectively. Let  $B'C'$  intersect  $AA'$  in  $P$  and  $AC$  in  $Q$ , and let  $BB'$  intersect  $AC$  in  $R$ . Suppose the quadrilateral  $PIRQ$  is a kite; that is,  $IP = IR$  and  $QP = QR$ . Prove that  $ABC$  is an equilateral triangle.
6. Show that there are infinitely many positive real numbers  $a$  which are not integers such that  $a(a - 3\{a\})$  is an integer. (Here  $\{a\}$  denotes the fractional part of  $a$ . For example  $\{1.5\} = 0.5$ ;  $\{-3.4\} = 0.6$ .)

## CRMO-2015 questions and solutions

1. Two circles  $\Gamma$  and  $\Sigma$ , with centres  $O$  and  $O'$ , respectively, are such that  $O'$  lies on  $\Gamma$ . Let  $A$  be a point on  $\Sigma$  and  $M$  the midpoint of the segment  $AO'$ . If  $B$  is a point on  $\Sigma$  different from  $A$  such that  $AB$  is parallel to  $OM$ , show that the midpoint of  $AB$  lies on  $\Gamma$ .

**Solution:** Let  $C$  be the reflection of  $O'$  with respect to  $O$ . Then in triangle  $O'AC$ , the midpoints of the segments  $O'A$  and  $O'C$  are  $M$  and  $O$ , respectively. This implies  $AC$  is parallel to  $OM$ , and hence  $B$  lies on  $AC$ . Let the line  $AC$  intersect  $\Gamma$  again at  $N$ . Since  $O'C$  is a diameter of  $\Gamma$  it follows that  $\angle O'NC = 90^\circ$ . Since  $O'A = O'B$ , we can now conclude that  $N$  is the midpoint of the segment  $AB$ .



2. Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial where  $a$  and  $b$  are real numbers. Suppose  $\langle P(-1)^2, P(0)^2, P(1)^2 \rangle$  is an arithmetic progression of integers. Prove that  $a$  and  $b$  are integers.

**Solution:** Observe that

$$P(-1) = 1 - a + b, \quad P(0) = b, \quad P(1) = 1 + a + b.$$

The given condition gives

$$2b^2 = (1 - a + b)^2 + (1 + a + b)^2 = 2(1 + b)^2 + 2a^2 = 2 + 4b + 2b^2 + 2a^2.$$

Hence  $a^2 + 2b + 1 = 0$ . Observe

$$1 + a^2 + b^2 + 2a + 2b + 2ab = (1 + a + b)^2 \in \mathbb{Z}.$$

But  $1, b^2, 2a^2 + 4b$  are all integers. Hence  $4a + 4ab \in \mathbb{Z}$ . This gives  $16a^2(1 + b)^2$  is an integer. But  $a^2 = -(2b + 1)$ . Hence  $16(2b + 1)(1 + b)^2$  is an integer. But

$$16(2b + 1)(1 + b)^2 = 16(1 + 4b + 5b^2 + 2b^3).$$

Hence  $16b(4 + 2b^2)$  is an integer. If  $b = 0$ , then  $b$  is an integer. Otherwise, this shows that  $b$  is a rational number. Because  $b^2 \in \mathbb{Z}$ , it follows that  $b$  is an integer. Since  $a^2 = -(2b + 1)$ , we get that  $a^2$  is an integer. Now  $4a(1 + b) \in \mathbb{Z}$ . If  $b \neq -1$ , then  $a$  is rational and hence  $a$  is an integer. If  $b = -1$ , then we see that  $P(-1) = -a$ ,  $P(0) = b = -1$  and  $P(1) = a$ . Hence  $a^2, 1, a^2$  is an AP. This implies that  $a^2 = 1$  and hence  $a = \pm 1$ .

3. Show that there are infinitely many triples  $(x, y, z)$  of integers such that  $x^3 + y^4 = z^{31}$ .

**Solution:** Choose  $x = 2^{4r}$  and  $y = 2^{3r}$ . Then the left side is  $2^{12r+1}$ . If we take  $z = 2^k$ , then we get  $2^{12r+1} = 2^{31k}$ . Thus it is sufficient to prove that the equation  $12r + 1 = 31k$  has infinitely many solutions in integers. Observe that  $(12 \times 18) + 1 = 31 \times 7$ . If we choose  $r = 31l + 18$  and  $k = 12l + 7$ , we get

$$12(31l + 18) + 1 = 31(12l + 7),$$

for all  $l$ . Choosing  $l \in \mathbb{N}$ , we get infinitely many  $r = 31l + 18$  and  $k = 12l + 7$  such that  $12r + 1 = 31k$ . Going back we have infinitely many  $(x, y, z)$  of integers satisfying the given equation.

4. Suppose 36 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

**Solution:** One can choose 3 objects out of 36 objects in  $\binom{36}{3}$  ways. Among these choices all would be together in 36 cases; exactly two will be together in  $36 \times 32$  cases. Thus three objects can be chosen such that no two adjacent in  $\binom{36}{3} - 36 - (36 \times 32)$  ways. Among these, further, two objects will be diametrically opposite in 18 ways and the third would be on either semicircle in a non adjacent portion in  $36 - 6 = 30$  ways. Thus required number is

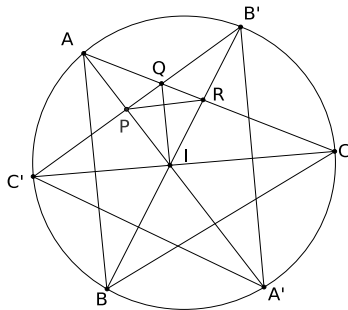
$$\binom{36}{3} - 36 - (36 \times 32) - (18 \times 30) = 5412.$$

5. Let  $ABC$  be a triangle with circumcircle  $\Gamma$  and incentre  $I$ . Let the internal angle bisectors of  $\angle A$ ,  $\angle B$  and  $\angle C$  meet  $\Gamma$  in  $A'$ ,  $B'$  and  $C'$  respectively. Let  $B'C'$  intersect  $AA'$  in  $P$  and  $AC$  in  $Q$ , and let  $BB'$  intersect  $AC$  in  $R$ . Suppose the quadrilateral  $PIRQ$  is a kite; that is,  $IP = IR$  and  $QP = QR$ . Prove that  $ABC$  is an equilateral triangle.

**Solution:** We first show that  $AA'$  is perpendicular to  $B'C'$ . Observe  $\angle C'A'A = \angle C'CA = \angle C/2$ ;  $\angle A'C'C = \angle A'AC = \angle A/2$ ; and  $\angle CC'B' = \angle CBB' = \angle B/2$ . Hence

$$\angle C'AP + \angle AC'P = \angle C'AB + \angle BAP + \angle AC'P = \frac{\angle C}{2} + \frac{\angle A}{2} + \frac{\angle B}{2} = 90^\circ.$$

It follows that  $\angle APC' = \angle A'PC' = 90^\circ$ . Thus  $\angle IPQ = 90^\circ$ . Since  $PIRQ$  is a kite, we observe that  $\angle IPR = \angle IRP$  and  $\angle QPR = \angle QRP$ . This implies that  $\angle IRQ = 90^\circ$ . Hence the kite  $IRQP$  is also a cyclic quadrilateral. Since  $\angle IRQ = 90^\circ$ , we see that  $BB' \perp AC$ . Since  $BB'$  is the bisector of  $\angle B$ , we conclude that  $\angle A = \angle C$ .



We also observe that the triangles  $IRC$  and  $IPB'$  are congruent triangles: they are similar, since  $\angle IRC = \angle IPB' = 90^\circ$  and  $\angle ICR = \angle C/2 = \angle IB'P (= \angle BCC')$ ; besides  $IR = IP$ . Therefore  $IC = IB'$ . But  $B'I = B'C$ . Thus  $IB'C$  is an equilateral triangle. This means  $\angle B'IC = 60^\circ$  and hence  $\angle ICR = 30^\circ$ . Therefore  $\angle C/2 = 30^\circ$ . Hence  $\angle A = \angle C = 60^\circ$ . It follows that  $ABC$  is equilateral.

6. Show that there are infinitely many positive real numbers  $a$  which are not integers such that  $a(a - 3\{a\})$  is an integer. (Here  $\{a\}$  denotes the fractional part of  $a$ . For example  $\{1.5\} = 0.5$ ;  $\{-3.4\} = 0.6$ .)

**Solution:** We show that for each integer  $n \geq 0$ , the interval  $(n, n+1)$  contains  $a$  such that  $a(a - 3\{a\})$  is an integer. Put  $a = n + f$ , where  $0 < f < 1$ . Then  $(n + f)(n - 2f)$  must be an integer. This means  $2f^2 + nf$  must be an integer. Since  $0 < f < 1$ , we must have  $0 < 2f^2 + nf < 2 + n$ . Hence  $2f^2 + nf \in \{1, 2, 3, \dots, n+1\}$ . Taking  $2f^2 + nf = 1$ , we get a quadratic equation:

$$2f^2 + nf - 1 = 0.$$

Hence

$$f = \frac{-n + \sqrt{n^2 + 8}}{4}, \text{ and } a = n + \frac{-n + \sqrt{n^2 + 8}}{4}.$$

Thus we see that each  $a$  in the set

$$\left\{ n + \frac{-n + \sqrt{n^2 + 8}}{4} : n \in \mathbb{N} \right\}$$

is a real number, which is not an integer, such that  $a(a - 3\{a\})$  is an integer.

**Remark:** Each interval  $(n, n+1)$  contains  $n+1$  such numbers, for  $n \geq 0$ ,  $n$  an integer.

————-000————-

# Regional Mathematical Olympiad-2015

Time: 3 hours

December 06, 2015

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be a triangle. Let  $B'$  denote the reflection of  $B$  in the internal angle bisector  $\ell$  of  $\angle A$ . Show that the circumcentre of the triangle  $CB'I$  lies on the line  $\ell$ , where  $I$  is the incentre of  $ABC$ .
2. Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial where  $a$  is real and  $b$  is rational. Suppose  $P(0)^2, P(1)^2, P(2)^2$  are integers. Prove that  $a$  and  $b$  are integers.
3. Find all integers  $a, b, c$  such that

$$a^2 = bc + 4, \quad b^2 = ca + 4.$$

4. Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?
5. Two circles  $\Gamma$  and  $\Sigma$  intersect at two distinct points  $A$  and  $B$ . A line through  $B$  intersects  $\Gamma$  and  $\Sigma$  again at  $C$  and  $D$ , respectively. Suppose that  $CA = CD$ . Show that the centre of  $\Sigma$  lies on  $\Gamma$ .
6. How many integers  $m$  satisfy both the following properties:  
(i)  $1 \leq m \leq 5000$ ; (ii)  $[\sqrt{m}] = [\sqrt{m+125}]$ ?  
(Here  $[x]$  denotes the largest integer not exceeding  $x$ , for any real number  $x$ .)

## CRMO-2015 questions and solutions

- Let  $ABC$  be a triangle. Let  $B'$  denote the reflection of  $B$  in the internal angle bisector  $\ell$  of  $\angle A$ . Show that the circumcentre of the triangle  $CB'I$  lies on the line  $\ell$ , where  $I$  is the incentre of  $ABC$ .

**Solution:** Let the line  $\ell$  meet the circumcircle of  $ABC$  in  $E$ . Then  $E$  is the midpoint of the minor arc  $BC$ . Hence  $EB = EC$ .

Note that  $\angle EBC = \angle EAC = A/2$  and  $\angle IBC = B/2$ . Hence

$$\angle BIE = \angle ABI + \angle BAI = B/2 + A/2.$$

We also have

$$\angle IBE = \angle IBC + \angle CBE = B/2 + A/2.$$

Therefore  $\angle BIE = \angle IBE$ , so that  $EB = EI$ . Since  $AE$  is the perpendicular bisector of  $BB'$ , we also have  $EB = EB'$ . Thus we get

$$EB' = EC = EI.$$

This implies that  $E$  is the circumcentre of  $\triangle CB'I$

- Let  $P(x) = x^2 + ax + b$  be a quadratic polynomial where  $a$  is real and  $b \neq 2$  is rational. Suppose  $P(0)^2, P(1)^2, P(2)^2$  are integers. Prove that  $a$  and  $b$  are integers.

**Solution:** We have  $P(0) = b$ . Since  $b$  is rational and  $b^2 = P(0)^2$  is an integer, we conclude that  $b$  is an integer. Observe that

$$\begin{aligned} P(1)^2 &= (1 + a + b)^2 = a^2 + 2a(1 + b) + (1 + b)^2 \in \mathbb{Z} \\ P(2)^2 &= (4 + 2a + b)^2 = 4a^2 + 4a(4 + b) + (4 + b)^2 \in \mathbb{Z} \end{aligned}$$

Eliminating  $a^2$ , we see that  $4a(b - 2) + 4(1 + b)^2 - (4 + b)^2 \in \mathbb{Z}$ . Since  $b \neq 2$ , it follows that  $a$  is rational. Hence the equation  $x^2 + 2x(1 + b) + (1 + b)^2 - (a^2 + 2a(1 + b) + (1 + b)^2) = 0$  is a quadratic equation with integer coefficients and has rational solution  $a$ . It follows that  $a$  is an integer.

- Find all integers  $a, b, c$  such that

$$a^2 = bc + 4, \quad b^2 = ca + 4.$$

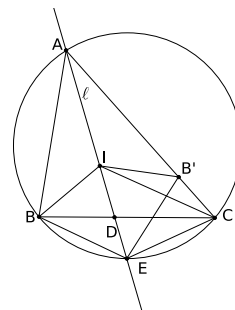
**Solution:** Suppose  $a = b$ . Then we get one equation:  $a^2 = ac + 4$ . This reduces to  $a(a - c) = 4$ . Therefore  $a = 1, a - c = 4$ ;  $a = -1, a - c = -4$ ;  $a = 4, a - c = 1$ ;  $a = -4, a - c = -1$ ;  $a = 2, a - c = 2$ ;  $a = -2, a - c = -2$ . Thus we get  $(a, b, c) = (1, 1, -3), (-1, -1, 3), (4, 4, 3), (-4, -4, -3); (2, 2, 0), (-2, -2, 0)$ .

If  $a \neq b$ , subtracting the second relation from the first we get

$$a^2 - b^2 = c(b - a).$$

This gives  $a + b = -c$ . Substituting this in the first equation, we get

$$a^2 = b(-a - b) + 4.$$



Thus  $a^2 + b^2 + ab = 4$ . Multiplication by 2 gives

$$(a + b)^2 + a^2 + b^2 = 8.$$

Thus  $(a, b) = (2, -2), (-2, 2), (2, 0), (-2, 0), (0, 2), (0, -2)$ . We get respectively  $c = 0, 0, -2, 2, -2, 2$ . Thus we get the triples:

$$(a, b, c) = (1, 1, -3), (-1, -1, 3), (4, 4, 3), (-4, -4, -3), (2, 2, 0), (-2, -2, 0), \\ (2, -2, 0), (-2, 2, 0), (2, 0, -2), (-2, 0, 2), (0, 2, -2), (0, -2, 2).$$

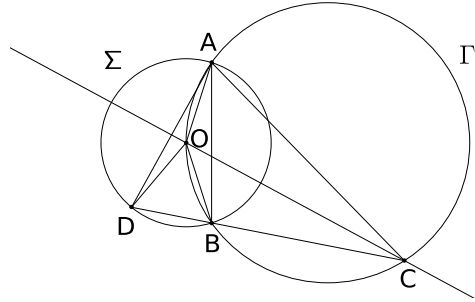
4. Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

**Solution:** One can choose 3 objects out of 40 objects in  $\binom{40}{3}$  ways. Among these choices all would be together in 40 cases; exactly two will be together in  $40 \times 36$  cases. Thus three objects can be chosen such that no two adjacent in  $\binom{40}{3} - 40 - (40 \times 36)$  ways. Among these, further, two objects will be diametrically opposite in 20 ways and the third would be on either semicircle in a non adjacent portion in  $40 - 6 = 34$  ways. Thus required number is

$$\binom{40}{3} - 40 - (40 \times 36) - (20 \times 34) = 7720.$$

5. Two circles  $\Gamma$  and  $\Sigma$  intersect at two distinct points  $A$  and  $B$ . A line through  $B$  intersects  $\Gamma$  and  $\Sigma$  again at  $C$  and  $D$ , respectively. Suppose that  $CA = CD$ . Show that the centre of  $\Sigma$  lies on  $\Gamma$ .

**Solution:** Let the perpendicular from  $C$  to  $AD$  intersect  $\Gamma$  at  $O$ . Since  $CA = CD$  we have that  $CO$  is the perpendicular bisector of  $AD$  and also the angular bisector of  $\angle ACD$ . From the former, it follows that  $OA = OD$ , and from the latter it follows that  $\angle OCB = \angle OCA$  and hence  $OA = OB$ . Thus we get  $OA = OB = OD$ . This means  $O$  is the circumcentre of triangle  $ADB$ . This shows that  $O$  is the centre of  $\Sigma$ .



6. How many integers  $m$  satisfy both the following properties:

$$(i) 1 \leq m \leq 5000; (ii) [\sqrt{m}] = [\sqrt{m+125}]?$$

(Here  $[x]$  denotes the largest integer not exceeding  $x$ , for any real number  $x$ .)

**Solution:** Let  $[\sqrt{m}] = [\sqrt{m+125}] = k$ . Then we know that

$$k^2 \leq m < m + 125 < (k + 1)^2.$$

Thus

$$m + 125 < k^2 + 2k + 1 \leq m + 2k + 1.$$

This shows that  $2k + 1 > 125$  or  $k > 62$ . Using  $k^2 \leq 5000$ , we get  $k \leq 70$ . Thus  $k \in \{63, 64, 65, 66, 67, 68, 69, 70\}$ . We observe that  $63^2 = 3969$  and  $64^2 = 63^2 + 127$ . Hence

$$[\sqrt{63^2 + 125}] = [\sqrt{63^2 + 1 + 125}] = 63,$$

but  $\lceil \sqrt{63^2 + 2 + 125} \rceil = 64$ . Thus we get two values of  $m$  such that  $\lceil \sqrt{m} \rceil = \lceil \sqrt{m + 125} \rceil$  for  $k = 63$ . Similarly,  $65^2 = 64^2 + 129$  so that

$$\lceil \sqrt{64^2 + 125} \rceil = \lceil \sqrt{64^2 + 1 + 125} \rceil = \lceil \sqrt{64^2 + 2 + 125} \rceil = \lceil \sqrt{64^2 + 3 + 125} \rceil = 64,$$

but  $\lceil \sqrt{64^2 + 4 + 125} \rceil = 65$ . Thus we get four values of  $m$  such that  $\lceil \sqrt{m} \rceil = \lceil \sqrt{m + 125} \rceil$  for  $k = 64$ . Continuing, we see that there are 6, 8, 10, 12, 14, 16 values of  $m$  respectively for  $k = 65, 66, 67, 68, 69, 70$ . Together we get

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 2 \times \frac{8 \times 9}{2} = 72$$

values of  $m$  satisfying the given requirement.

————-0000————-

# Regional Mathematical Olympiad 2015 (Mumbai region)

## 06 December, 2015

- There are eight questions in this question paper. Answer all questions.
- Each of the questions 1,2,3 carries 10 points. Each of the questions 4,5,6,7,8 carries 14 points.
- Use of protractors, calculators, mobile phone is forbidden.
- Time allotted: 4 hours

1. Let  $ABCD$  be a convex quadrilateral with  $AB = a$ ,  $BC = b$ ,  $CD = c$  and  $DA = d$ . Suppose

$$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da,$$

and the area of  $ABCD$  is 60 square units. If the length of one of the diagonals is 30 units, determine the length of the other diagonal.

2. Determine the number of 3-digit numbers in base 10 having at least one 5 and at most one 3.
3. Let  $P(x)$  be a non-constant polynomial whose coefficients are positive integers. If  $P(n)$  divides  $P(P(n) - 2015)$  for every natural number  $n$ , prove that  $P(-2015) = 0$ .
4. Find all three digit natural numbers of the form  $(abc)_{10}$  such that  $(abc)_{10}$ ,  $(bca)_{10}$  and  $(cab)_{10}$  are in geometric progression. (Here  $(abc)_{10}$  is representation in base 10.)
5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$  and let  $BD$  be the altitude from  $B$  on to  $AC$ . Draw  $DE \perp AB$  and  $DF \perp BC$ . Let  $P$ ,  $Q$ ,  $R$  and  $S$  be respectively the incentres of triangle  $DFC$ ,  $DBF$ ,  $DEB$  and  $DAE$ . Suppose  $S$ ,  $R$ ,  $Q$  are collinear. Prove that  $P$ ,  $Q$ ,  $R$ ,  $D$  lie on a circle.
6. Let  $S = \{1, 2, \dots, n\}$  and let  $T$  be the set of all ordered triples of subsets of  $S$ , say  $(A_1, A_2, A_3)$ , such that  $A_1 \cup A_2 \cup A_3 = S$ . Determine, in terms of  $n$ ,

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3|$$

where  $|X|$  denotes the number of elements in the set  $X$ . (For example, if  $S = \{1, 2, 3\}$  and  $A_1 = \{1, 2\}$ ,  $A_2 = \{2, 3\}$ ,  $A_3 = \{3\}$  then one of the elements of  $T$  is  $(\{1, 2\}, \{2, 3\}, \{3\})$ .)

7. Let  $x, y, z$  be real numbers such that  $x^2 + y^2 + z^2 - 2xyz = 1$ . Prove that

$$(1+x)(1+y)(1+z) \leq 4 + 4xyz.$$

8. The length of each side of a convex quadrilateral  $ABCD$  is a positive integer. If the sum of the lengths of any three sides is divisible by the length of the remaining side then prove that some two sides of the quadrilateral have the same length.

**END OF QUESTION PAPER**

# Regional Mathematical Olympiad 2015 (Mumbai region)

## 06 December, 2015

### Hints and Solutions

1. Let  $ABCD$  be a convex quadrilateral with  $AB = a$ ,  $BC = b$ ,  $CD = c$  and  $DA = d$ . Suppose

$$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da,$$

and the area of  $ABCD$  is 60 square units. If the length of one of the diagonals is 30 units, determine the length of the other diagonal.

### Solution

$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da \Rightarrow (a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 = 0 \Rightarrow a = b = c = d$ .  
Thus  $ABCD$  is a rhombus and

$$[ABCD] = (1/2)(d_1 d_2) \tag{1}$$

where  $d_1$  and  $d_2$  are the lengths of the diagonals. Hence  $d_2 = \frac{2[ABCD]}{d_1} = 4$  units.

2. Determine the number of 3-digit numbers in base 10 having at least one 5 and at most one 3.

### Solution

We count the number of 3-digit numbers with (i) at least one 5 and having no 3 and (ii) at least one 5 and having exactly one 3 separately.

(i) Here we first count the whole set and subtract the number of 3-digit numbers having no 5 from it. Since 3 is not there and 0 cannot be the first digit, we can fill the first digit in 8 ways. But we can fill the second and third digits in 9 ways (as 0 can be included). Thus we get  $8 \times 9 \times 9$  such numbers. If no 5 is there, then the number of such numbers is  $7 \times 8 \times 8$ . Thus the number of 3-digit numbers not containing 3 and having at least one 5 is  $(8 \times 9 \times 9) - (7 \times 8 \times 8) = 8(81 - 56) = 200$ .

(ii) If 3 is there as a digit, then it can be the first digit or may be the second or third digit. Consider those numbers in which 3 is the first digit. The number of such numbers having at least one 5 is  $(9 \times 9) - (8 \times 8) = 81 - 64 = 17$ . The number of 3-digit numbers in which the second digit is 3 and having at least one 5 is  $(8 \times 9) - (7 \times 8) = 16$ . Similarly, the number of 3-digit numbers in which the third digit is 3 and having at least one 5 is  $(8 \times 9) - (7 \times 8) = 16$ . Thus we get  $17 + 16 + 16 = 49$  such numbers.

Therefore the number of 3-digit numbers having at most one 3 and at least one 5 is  $200 + 49 = 249$ .

3. Let  $P(x)$  be a non-constant polynomial whose coefficients are positive integers. If  $P(n)$  divides  $P(P(n) - 2015)$  for every natural number  $n$ , prove that  $P(-2015) = 0$ .

### Solution

Note that  $P(n) - 2015 - (-2015) = P(n)$  divides  $P(P(n) - 2015) - P(-2015)$  for every positive integer  $n$ . But  $P(n)$  divides  $P(P(n) - 2015)$  for every positive integer  $n$ . Therefore  $P(n)$  divides  $P(-2015)$  for every positive integer  $n$ . Hence  $P(-2015) = 0$ .

### Note

In the original version of the problem the word ‘non-constant’ was missing. The falsity of the statement was brought to the attention of the examiners by a contestant who mentioned it in a remark at the end of a perfect solution to the problem assuming that the polynomial is non-constant. Many students assumed that the polynomial is non-constant and completed the solution. They deserved full credit for doing so.

4. Find all three digit natural numbers of the form  $(abc)_{10}$  such that  $(abc)_{10}$ ,  $(bca)_{10}$  and  $(cab)_{10}$  are in geometric progression. (Here  $(abc)_{10}$  is representation in base 10.)

### Solution

Let us write

$$x = (a \times 10^2) + (b \times 10) + c, \quad y = (b \times 10^2) + (c \times 10) + a, \quad z = (c \times 10^2) + (a \times 10) + b.$$

We are given that  $y^2 = xz$ . This means

$$((b \times 10^2) + (c \times 10) + a)^2 = ((a \times 10^2) + (b \times 10) + c)((c \times 10^2) + (a \times 10) + b).$$

We can solve for  $c$  and get

$$c = \frac{10b^2 - a^2}{10a - b}.$$

If  $a, b, c$  are digits leading to a solution, and if  $d = \gcd(a, b)$  then  $d|c$ . Consequently, we may assume that  $\gcd(a, b) = 1$ . Now

$$c = \frac{999a^2}{10a - b} - (10b + 100a),$$

showing that  $10a - b$  divides  $999a^2$ . Since  $a, b$  are relatively prime, this is possible only if  $10a - b$  is a factor of 999. It follows that  $10a - b$  takes the values 1, 3, 9, 27, 37. These values lead to the pairs

$$(a, b) = (1, 9), (1, 7), (1, 1), (4, 3).$$

We can discard the first two pairs as they lead to a value of  $c > 10$ . The third gives the trivial solution (111, 111, 111). Taking  $d = 2, 3, 4, 5, 6, 7, 8, 9$ , we get 9 solution:

$$(abc)_{10} = 111, 222, 333, 444, 555, 666, 777, 888, 999.$$

The last pair gives  $c = 2$  and hence the solution (432, 324, 243). Another solution is obtained on multiplying by 2: (864, 648, 486).

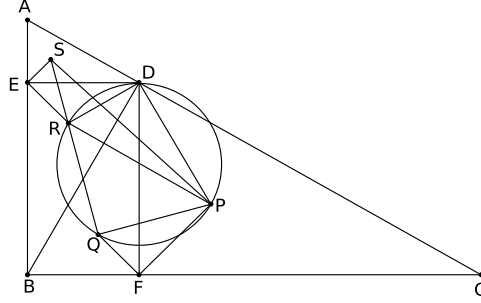
Thus we have

$$(abc)_{10} = 111, 222, 333, 444, 555, 666, 777, 888, 999, 432, 864.$$

5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$  and let  $BD$  be the altitude from  $B$  on to  $AC$ . Draw  $DE \perp AB$  and  $DF \perp BC$ . Let  $P, Q, R$  and  $S$  be respectively the incentres of triangle  $DFC, DBF, DEB$  and  $DAE$ . Suppose  $S, R, Q$  are collinear. Prove that  $P, Q, R, D$  lie on a circle.

### Solution

We first show that  $SR$  is perpendicular to  $QP$ . Consider triangles  $PFQ$  and  $RES$ . Observe that  $AE \parallel DF$  and  $ED \parallel FC$ . Since  $ES$  bisects  $\angle AED$  and  $FP$  bisects  $\angle DFC$ , it follows that  $ES \parallel FP$ . Since  $ER \perp ES$  and  $FQ \perp FP$ , we also have  $ER \parallel FQ$ .



If  $r_1$  and  $r_2$  are inradii of triangles  $DEA$  and  $DEB$ , and if  $r'$  is the inradius of  $\triangle DAB$ , we know that

$$r_1 = r' \frac{AD}{AB}, \quad r_2 = r' \frac{BD}{AB}.$$

Hence

$$\frac{ES}{ER} = \frac{r_1}{r_2} = \frac{AD}{BD}.$$

Similarly, we can prove that

$$\frac{FQ}{FP} = \frac{BD}{DC}.$$

But we know that  $AD/BD = BD/DC$ . Hence we conclude that  $ES/ER = FQ/FP$ . Therefore  $\triangle ESR \sim \triangle QFP$ . But  $SE \perp ER$  and  $ER \parallel QF$  imply  $SE \perp QF$ . It follows that  $SR \perp QP$ .

Since  $S, R, Q$  are collinear, we get  $SQ \perp QP$ . Thus  $\angle RQP = 90^\circ$ . Consider the circumcircle of  $\triangle PRQ$ . Since  $\angle RDP = 90^\circ$ , we conclude that  $D$  lies on this circle. Hence  $P, Q, R, D$  are concyclic.

6. Let  $S = \{1, 2, \dots, n\}$  and let  $T$  be the set of all ordered triples of subsets of  $S$ , say  $(A_1, A_2, A_3)$ , such that  $A_1 \cup A_2 \cup A_3 = S$ . Determine, in terms of  $n$ ,

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3|$$

where  $|X|$  denotes the number of elements in the set  $X$ . (For example, if  $S = \{1, 2, 3\}$  and  $A_1 = \{1, 2\}$ ,  $A_2 = \{2, 3\}$ ,  $A_3 = \{3\}$  then one of the elements of  $T$  is  $(\{1, 2\}, \{2, 3\}, \{3\})$ .)

### Solution 1

Let  $X = (A_1, A_2, A_3) \in T$  and let  $i \in A_1 \cap A_2 \cap A_3$ . The number of times the element  $i$  occurs in the required sum is equal to the number of ordered tuples  $(A_1 - \{i\}, A_2 - \{i\}, A_3 - \{i\})$  such that

$$A_1 - \{i\} \cup A_2 - \{i\} \cup A_3 - \{i\} = S - \{i\} \quad (*)$$

For every element of  $S - \{i\}$ , there are eight possibilities - whether the element belongs to or does not belong to  $A_i$  for  $i = 1, 2, 3$ . Out of these the case when the element does not belong to any of the three subsets violates (\*). Therefore, each element can satisfy the requirement in 7 ways. The number of tuples is, therefore,  $7^{n-1}$  and the sum is  $n \cdot 7^{n-1}$ .

### Solution 2

Consider all tuples  $(A_1, A_2, A_3)$  such that  $A_1 \cap A_2 \cap A_3 = B$  (\*) and  $A_1 \cup A_2 \cup A_3 = S$  (\*\*). Let  $|B| = r$ . (\*) and (\*\*) lead to

$$A_1 - B \cap A_2 - B \cap A_3 - B = \Phi \quad (***) \text{ and } A_1 - B \cup A_2 - B \cup A_3 - B = S - B \quad (****)$$

As before, for every element of  $S - B$ , there are eight possibilities. Out of these two cases -i. the element belongs to each of the three subsets  $A_1, A_2, A_3$  violates (\*\*\*) and ii. the element does not belong to any of the three subsets violates (\*\*\*\*). Therefore, there are 6 possibilities for each element. Also,  $|S - B| = n - r$ , therefore, the number of tuples satisfying (\*) and (\*\*) is  $6^{n-r}$ . The number of ways we can select  $r$  elements is  $\binom{n}{r}$ , therefore the required number is

$$\sum_{r=0}^n \binom{n}{r} r 6^{n-r} = n 7^{n-1}$$

### Solution 3

**This solution was given by a contestant of standard IX. We provide a sketch of the same.**

Write  $S_n$  and  $T_n$  in place of  $S$  and  $T$  respectively. Also, let

$$\sum_{(A_1, A_2, A_3) \in T_n} |A_1 \cap A_2 \cap A_3| = a_n.$$

We shall show that  $a_n = n \cdot 7^{n-1}$  using recurrence and induction. Let  $a_n = n \cdot 7^{n-1}$  for some positive integer  $n$ . Then, the elements of  $T_{n+1}$  are made by adding  $n+1$  to either  $A_1$  or  $A_2$  or  $A_3$  or  $A_1 \cap A_2$  or  $A_2 \cap A_3$  or  $A_3 \cap A_1$  or  $A_1 \cap A_2 \cap A_3$  in  $T_n$ . Further, it is easily established that  $|T_n| = 7^n$ . Now, if  $n+1$  is added to  $A_1 \cap A_2 \cap A_3$  then  $|A_1 \cap A_2 \cap A_3|$  increases by 1. Otherwise it remains the same. So, as there are seven choices of where to put  $n+1$ ,

$$a_{n+1} = 7a_n + |T_n|$$

Thus  $a_{n+1} = (n+1) \cdot 7^n$ , which completes the inductive step, as the base case  $n = 1$  is trivial. Therefore

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3| = n \cdot 7^{n-1}$$

7. Let  $x, y, z$  be real numbers such that  $x^2 + y^2 + z^2 - 2xyz = 1$ . Prove that

$$(1+x)(1+y)(1+z) \leq 4 + 4xyz.$$

### Solution

Write  $1 + 2xyz = x^2 + y^2 + z^2 \Leftrightarrow 3 + 3xyz = \frac{3}{2}(x^2 + y^2 + z^2) + \frac{3}{2}$   
 $\Rightarrow 3 + 3xyz = (x^2 + y^2 + z^2) + \frac{1}{2}[(x^2 + 1) + (y^2 + 1) + (z^2 + 1)] \geq x^2 + y^2 + z^2 + x + y + z$   
(Use  $x^2 + y^2 + z^2 \geq xy + yz + zx$  and AM-GM:  $x^2 + 1 \geq 2x$  etc.)  
 $\Rightarrow 3 + 3xyz \geq xy + yz + zx + x + y + z$ .  
By adding  $1 + xyz$  in both sides,  
we get  $4 + 4xyz \geq 1 + x + y + z + xy + yz + zx + xyz = (1 + x)(1 + y)(1 + z)$ .  
Equality holds when  $x = y = z = 1$ .

8. The length of each side of a convex quadrilateral  $ABCD$  is a positive integer. If the sum of the lengths of any three sides is divisible by the length of the remaining side then prove that some two sides of the quadrilateral have the same length.

### Solution

Let us take  $ABCD$  to be a quadrilateral with  $AB = a$ ,  $BC = b$ ,  $CD = c$  and  $DA = d$  where  $a, b, c, d$  are integers. We know  $AC < AB + BC$  and  $AD < AC + CD < AB + BC + CD$ . Thus we get  $d < a + b + c$ . Similarly we can write down three more inequalities:  $a < b + c + d$ ,  $b < c + d + a$  and  $c < d + a + b$ . We can also take  $d$  to be the largest side. Using the given conditions, we can write

$$a + b + c + d = la, \quad b + c + d + a = mb, \quad c + d + a + b = nc, \quad a + b + c + d = kd,$$

for some positive integers  $l, m, n, k$ .

Suppose no two sides are equal. Then  $a < d$ ,  $b < d$ ,  $c < d$ . Hence  $a + b + c < 3d$ . Since  $d$  divides  $a + b + c$  and  $d < a + b + c < 3d$ , we must have  $a + b + c = 2d$ . Thus we obtain

$$a + b + c + d = 3d = la = mb = nc.$$

Write this as

$$a = \frac{3d}{l}, \quad b = \frac{3d}{m}, \quad c = \frac{3d}{n}.$$

Using  $2d = a + b + c$ , we get the equation

$$\frac{3}{l} + \frac{3}{m} + \frac{3}{n} = 2.$$

Here  $l, m, n$  are necessarily distinct. Suppose  $l = m$ . Then  $la = 3d = lb$ . This implies  $a = b$ , a contradiction. Similarly  $m = n$  and  $l = n$  can be discarded. We may assume  $l < m < n$ . This means, we have to solve the equation for distinct positive integers.

If  $l \geq 4$ , then  $m \geq 5$  and  $n \geq 6$ . Hence

$$\frac{2}{3} = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} \leq \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60} < \frac{2}{3}$$

which is impossible. This means  $l = 1$ ,  $l = 2$  or  $l = 3$ . For  $l = 1$ , we get  $b + c + d = 0$ , Which is impossible. If  $l = 2$ , we get  $b + c + d = a$ , which contradicts  $a < b + c + d$ . If  $l = 3$ , then  $a = d$  and this contradicts again  $a < d$ . Therefore some two of  $a, b, c, d$  are equal.

**THE END**

### NOTE

We do not claim that the solutions presented here are the most elegant solutions but we thought they would be instructive. For some problems we found that solutions by contestants were different and we have included them in this document.

# Regional Mathematical Olympiad-2016

Time: 3 hours

October 09, 2016

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be the incentre of  $ABC$ . Draw a line perpendicular to  $AI$  at  $I$ . Let it intersect the line  $CB$  at  $D$ . Prove that  $CI$  is perpendicular to  $AD$  and prove that  $ID = \sqrt{b(b-a)}$  where  $BC = a$  and  $CA = b$ .

2. Let  $a, b, c$  be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that  $abc \leq 1/8$ .

3. For any natural number  $n$ , expressed in base 10, let  $S(n)$  denote the sum of all digits of  $n$ . Find all natural numbers  $n$  such that  $n = 2S(n)^2$ .
4. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.
5. Let  $ABC$  be a triangle with centroid  $G$ . Let the circumcircle of triangle  $AGB$  intersect the line  $BC$  in  $X$  different from  $B$ ; and the circumcircle of triangle  $AGC$  intersect the line  $BC$  in  $Y$  different from  $C$ . Prove that  $G$  is the centroid of triangle  $AXY$ .
6. Let  $\langle a_1, a_2, a_3, \dots \rangle$  be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

1. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be the incentre of  $ABC$ . Draw a line perpendicular to  $AI$  at  $I$ . Let it intersect the line  $CB$  at  $D$ . Prove that  $CI$  is perpendicular to  $AD$  and prove that  $ID = \sqrt{b(b-a)}$  where  $BC = a$  and  $CA = b$ .

**Solution:** First observe that  $ADBI$  is a cyclic quadrilateral since  $\angle AID = \angle ABD = 90^\circ$ . Hence  $\angle ADI = \angle ABI = 45^\circ$ . Hence  $\angle DAI = 45^\circ$ . But we also have

$$\begin{aligned}\angle ADB &= \angle ADI + \angle IDB = 45^\circ + \angle IAB \\ &= \angle DAI + \angle IAC = \angle DAC.\end{aligned}$$

Therefore  $CDA$  is an isosceles triangle with  $CD = CA$ . Since  $CI$  bisects  $\angle C$  it follows that  $CI \perp AD$ .

This shows that  $DB = CA - CB = b - a$ . Therefore

$$AD^2 = c^2 + (b-a)^2 = c^2 + b^2 + a^2 - 2ba = 2b(b-a).$$

But then  $2ID^2 = AD^2 = 2b(b-a)$  and this gives  $ID = \sqrt{b(b-a)}$ .

2. Let  $a, b, c$  be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that  $abc \leq 1/8$ .

**Solution:** This is equivalent to

$$\sum a(1+b)(1+c) = (1+a)(1+b)(1+c).$$

This simplifies to

$$ab + bc + ca + 2abc = 1$$

Using AM-GM inequality, we have

$$1 = ab + bc + ca + 2abc \geq 4(ab \cdot bc \cdot ca \cdot 2abc)^{1/4}.$$

Simplification gives

$$abc \leq \frac{1}{8}.$$

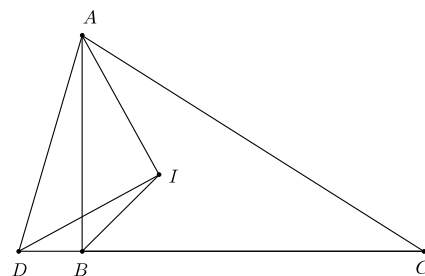
3. For any natural number  $n$ , expressed in base 10, let  $S(n)$  denote the sum of all digits of  $n$ . Find all natural numbers  $n$  such that  $n = 2S(n)^2$ .

**Solution:** We use the fact that 9 divides  $n - S(n)$  for every natural number  $n$ . Hence  $S(n)(2S(n)-1)$  is divisible by 9. Since  $S(n)$  and  $2S(n)-1$  are relatively prime, it follows that 9 divides either  $S(n)$  or  $2S(n)-1$ , but not both. We also observe that the number of digits of  $n$  cannot exceed 4. If  $n$  has  $k$  digits, then  $n \geq 10^{k-1}$  and  $2S(n)^2 \leq 2 \times (9k)^2 = 162k^2$ . If  $k \geq 6$ , we see that

$$2S(n)^2 \leq 162k^2 < 5^4 k^2 < 10^{k-1} \leq n.$$

If  $k = 5$ , we have

$$2S(n)^2 \leq 162 \times 25 = 4150 < 10^4 \leq n.$$



Therefore  $n \leq 4$  and  $S(n) \leq 36$ .

If  $9|S(n)$ , then  $S(n) = 9, 18, 27, 36$ . We see that  $2S(n)^2$  is respectively equal to 162, 648, 1458, 2592. Only 162 and 648 satisfy  $n = 2S(n)^2$ .

If  $9|(2S(n) - 1)$ , then  $2S(n) = 9k + 1$ . Only  $k = 1, 3, 5, 7$  give integer values for  $S(n)$ . In these cases  $2S(n)^2 = 50, 392, 1058, 2048$ . Here again 50 and 392 give  $n = 2S(n)^2$ .

Thus the only natural numbers with the property  $n = 2S(n)^2$  are 50, 162, 392, 648.

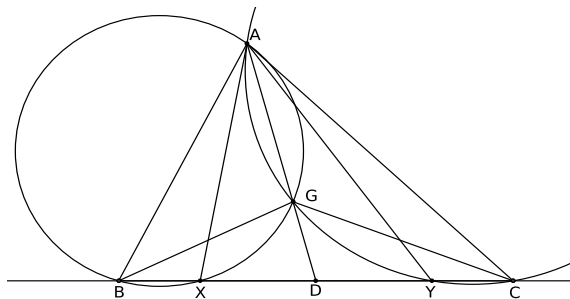
4. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

**Solution:** First we choose 3 places for even digits. This can be done in  $\binom{6}{3} = 20$  ways. Observe that the other places for odd digits get automatically fixed. There are 5 even digits and 5 odd digits. Any of these can occur in their proper places. Hence there are  $5^3$  ways of selecting 3 even and 3 odd digits for a particular selection of place for even digits. Hence we get  $20 \times 5^3$  such numbers. But this includes all those numbers having the first digit equal to 0. Since we are looking for 6-digit numbers, these numbers have to be removed from our counting. If we fix 0 as the first digit, we have, 2 places for even numbers and 3 places for odd numbers. We can choose 2 places for even numbers in  $\binom{5}{2} = 10$  ways. As earlier, for any such choice of places for even digits, we can choose even digits in  $5^2$  ways and odd digits in  $5^3$  ways. Hence the number of ways of choosing 3 even and 3 odd digits with 0 as the first digit is  $10 \times 5^5$ . Therefore the number of 6-digit numbers with 3 even digits and 3 odd digits is

$$20 \times 5^3 - 10 \times 5^5 = 10 \times 5^5(10 - 1) = 281250.$$

5. Let  $ABC$  be a triangle with centroid  $G$ . Let the circumcircles of  $\triangle AGB$  and  $\triangle AGC$  intersect the line  $BC$  in  $X$  and  $Y$  respectively, which are distinct from  $B, C$ . Prove that  $G$  is the centroid of  $\triangle AXY$ .

**Solution:** Let  $D$  be the midpoint of  $AB$ . Observe that  $DX \cdot DB = DG \cdot DA = DY \cdot DC$ . But  $DB = DC$ . Hence  $DX = DY$ . This means that  $D$  is the midpoint of  $XY$  as well. Hence  $AD$  is also a median of  $\triangle AXY$ . Now we know that  $AG : GD = 2 : 1$ . If  $G'$  is the median of  $\triangle AXY$ , then  $G'$  must lie on  $AD$  and  $AG' : G'D = 2 : 1$ . We conclude that  $G = G'$ .



6. Let  $\langle a_1, a_2, a_3, \dots \rangle$  be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

**Solution:** Let  $\langle a_1, a_2, \dots, a_{n+1}, \dots \rangle = \langle a, a + d, \dots, a + nd, \dots \rangle$  be a strictly increasing sequence of positive integers in arithmetic progression. Here  $a$  and  $d$  are both positive integers. Consider the following subsequence:

$$\langle a, a(1 + d), a(1 + d)^2, \dots, a(1 + d)^n, \dots \rangle.$$

This is a geometric progression. Here  $a > 0$  and the common ratio  $1 + d > 1$ . Hence the sequence is strictly increasing. The first term is  $a$  which is in the given AP. The second term is  $a(1 + d) = a + ad$  which is the  $(a + 1)$ -th term of the AP. In general, we see that

$$a(1 + d)^n = a + d \left( \binom{n}{1}a + \binom{n}{2}ad + \dots + \binom{n}{n}ad^{n-1} \right).$$

Here the coefficient of  $d$  in the braces is also a positive integer. Hence  $a(1 + d)^n$  is also a term of the given AP.

# Regional Mathematical Olympiad-2016

Time: 3 hours

October 09, 2016

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be the incentre of  $ABC$ . Let  $AI$  extended intersect  $BC$  at  $F$ . Draw a line perpendicular to  $AI$  at  $I$ . Let it intersect  $AC$  at  $E$ . Prove that  $IE = IF$ .
2. Let  $a, b, c$  be positive real numbers such that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} = 1.$$

Prove that  $abc \leq 1/8$ .

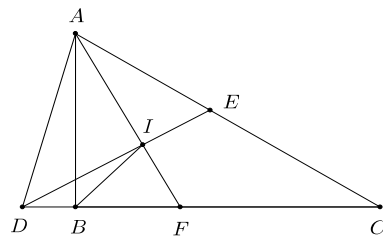
3. For any natural number  $n$ , expressed in base 10, let  $S(n)$  denote the sum of all digits of  $n$ . Find all natural numbers  $n$  such that  $n^3 = 8S(n)^3 + 6nS(n) + 1$ .
4. How many 6-digit natural numbers containing only the digits 1,2,3 are there in which 3 occurs exactly twice and the number is divisible by 9?
5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $AD$  be the bisector of  $\angle A$  with  $D$  on  $BC$ . Let the circumcircle of triangle  $ACD$  intersect  $AB$  again in  $E$ ; and let the circumcircle of triangle  $ABD$  intersect  $AC$  again in  $F$ . Let  $K$  be the reflection of  $E$  in the line  $BC$ . Prove that  $FK = BC$ .
6. Show that the infinite arithmetic progression  $\langle 1, 4, 7, 10, \dots \rangle$  has infinitely many 3-term subsequences in harmonic progression such that for any two such triples  $\langle a_1, a_2, a_3 \rangle$  and  $\langle b_1, b_2, b_3 \rangle$  in harmonic progression, one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

1. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be the incentre of  $ABC$ . Let  $AI$  extended intersect  $BC$  in  $F$ . Draw a line perpendicular to  $AI$  at  $I$ . Let it intersect  $AC$  in  $E$ . Prove that  $IE = IF$ .

**Solution:** Extend  $EI$  to meet  $CB$  extended in  $D$ . First observe that  $ADBI$  is a cyclic quadrilateral since  $\angle AID = \angle ABD$ . Hence  $\angle ADI = \angle ABI = 45^\circ$ . Hence  $\angle DAI = 45^\circ$ . Therefore  $IA = ID$ .

Consider the triangles  $AIE$  and  $DIF$ . Both are right triangles. Moreover  $\angle IAE = \angle IAB = \angle IDB$ . Since  $IA = ID$ , the triangles are congruent. This means  $IE = IF$ .



2. Let  $a, b, c$  be positive real numbers such that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} = 1.$$

Prove that  $abc \leq 1/8$ .

**Solution:** This is equivalent to

$$\sum a(1+c)(1+a) = (1+a)(1+b)(1+c).$$

This simplifies to

$$\sum a^2 + \sum a^2c = 1 + abc$$

Using AM-GM inequality, we have

$$1 + abc = \sum a^2 + \sum a^2c \geq 3(abc)^{2/3} + 3abc.$$

Let  $x = (abc)^{1/3}$ . Then

$$3x^2 + 2x^3 \leq 1.$$

This can be written as  $(x+1)^2(2x-1) \leq 0$ . Hence  $x \leq 1/2$ . Thus

$$abc \leq \frac{1}{8}.$$

3. For any natural number  $n$ , expressed in base 10, let  $S(n)$  denote the sum of all digits of  $n$ . Find all natural numbers  $n$  such that  $n^3 = 8S(n)^3 + 6nS(n) + 1$ .

**Solution:** We write the given condition as

$$n^3 + (-2S(n))^3 + (-1)^3 = 3 \times n \times (-2S(n)) \times (-1).$$

This is in the form  $x^3 + y^3 + z^3 = 3xyz$ . We know that this can happen if and only if  $x + y + z = 0$ . Thus we obtain a simpler condition

$$n - 2S(n) - 1 = 0.$$

Again we know that  $n - S(n)$  is divisible by 9. Hence 9 should divide  $S(n) + 1$ . It is easy to see that the number of digits in  $n$  cannot be more than 2. For a three digit number maximum value of  $S(n)$  can be 27 and  $2S(n) + 1 \leq 55$ . Hence  $n$  is either a 1-digit number or a two digit number. Hence  $S(n) \leq 18$ . Since 9 divides  $S(n) + 1$ , we can have  $S(n) = 8$  or  $S(n) = 17$ . But then  $n = 17$  or  $n = 35$ . Among these  $n = 17$  works but not 35. ( $S(35) = 8$  and  $2S(n) + 1 = 17 \neq 35$ .) Hence the only solution is  $n = 17$ .

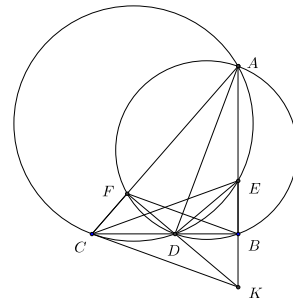
4. How many 6-digit natural numbers containing only the digits 1, 2, 3 are there in which 3 occurs exactly twice and the number is divisible by 9?

**Solution:**

Let  $S(n)$  be the sum of the digits of  $n$ . Then  $n \equiv S(n) \pmod{9}$ . For any admissible  $n$  we observe that  $10 \leq S(n) \leq 14$  and hence there is no value of  $S(n)$  that is a multiple of 9. Thus no such  $n$  exists.

5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $AD$  be the bisector of  $\angle A$  with  $D$  on  $BC$ . Let the circumcircle of triangle  $ACD$  intersect  $AB$  again in  $E$ ; and let the circumcircle of triangle  $ABD$  intersect  $AC$  again in  $F$ . Let  $K$  be the reflection of  $E$  in the line  $BC$ . Prove that  $FK = BC$ .

**Solution:** First we show that  $EB = FC$ . Consider the triangles  $EBD$  and  $CFD$ . Observe that  $\angle CFD = 90^\circ$  since  $\angle AFD = 90^\circ$  (angle in a semi-circle). Hence  $\angle CFD = \angle EBD$ . Since  $ACDE$  is a cyclic quadrilateral, we have  $\angle CDE = 180^\circ - \angle A$ . Similarly, we see that  $AFDB$  is a cyclic quadrilateral and therefore  $\angle FDB = 180^\circ - \angle A$ . Thus we obtain  $\angle CDE = \angle FDB$ . This gives  $\angle FDC = \angle BDE$ . It follows  $\triangle EBD \sim \triangle CFD$ .



Since  $AD$  bisects  $\angle A$ , we have  $DB = DF$ . Hence  $\triangle EBD \cong \triangle CFD$ . Hence  $FC = EB = BK$ . We also observe that  $AF = AB$  since  $\triangle ABD \cong \triangle AFD$ . It follows that  $FB \parallel CK$ . Since  $FC = BK$ , we conclude that  $CKDF$  is an isosceles trapezium. This gives  $FK = BC$ .

**Alternate solution:** First we show that  $K, D, F$  are collinear. Observe that  $\angle FDB = 180^\circ - \angle A$  by the concyclicity of  $AFDB$ . Moreover  $\angle BDK = \angle BDE = \angle A$ . Therefore  $\angle KDF = 180^\circ$ . This proves that  $KDF$  is a line segment.

Consider the triangles  $AKF$  and  $ABC$ . Since both are right-angled triangles and  $\angle A$  is common, they are similar. We also see that  $\triangle AFD \cong \triangle ABD$  since  $\angle AFD = \angle ABD = 90^\circ$ ,  $\angle FAD = \angle BAD = \angle A/2$  and  $AD$  common. Hence  $AF = AB$ . This implies now that  $\triangle AFK \cong \triangle ABC$ . Hence  $KF = BC$ .

6. Show that the infinite arithmetic progression  $\langle 1, 4, 7, 10, \dots \rangle$  has infinitely many 3-term subsequences in harmonic progression such that for any two such triples  $\langle a_1, a_2, a_3 \rangle$  and  $\langle b_1, b_2, b_3 \rangle$  in harmonic progression, one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2} \left( \frac{a_2}{b_2} \neq \frac{a_3}{b_3} \right).$$

**Solution:** Consider  $\langle 4, 7, 28 \rangle$ . We observe that

$$\frac{1}{4} + \frac{1}{28} = \frac{2}{7}.$$

Thus we look for the terms of the form  $a, b, ab$  which give a HP. The condition is

$$\frac{1}{a} + \frac{1}{ab} = \frac{2}{b}.$$

This reduces to  $b(1 + b) = 2ab$  or  $2a = 1 + b$ . The terms of the given AP are of the form  $3k + 1$ . If we take  $a = 3k + 1$ , then  $b = 2a - 1 = 6k + 1$ . We observe that  $b$  is also a term of the given AP.

Besides,  $ab = (3k + 1)(6k + 1) = 3(6k^2 + 3k) + 1$  is again a term of the given AP. Thus the triple of the form  $\langle 3k + 1, 6k + 1, (3k + 1)(6k + 1) \rangle$  form a HP. We observe that

$$\frac{3k + 1}{3l + 1} \neq \frac{6k + 1}{6l + 1} \neq \frac{(3k + 1)(6k + 1)}{(3l + 1)(6l + 1)}.$$

—————0—————

# Regional Mathematical Olympiad-2016

Time: 3 hours

October 16, 2016

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be a triangle and  $D$  be the mid-point of  $BC$ . Suppose the angle bisector of  $\angle ADC$  is tangent to the circumcircle of triangle  $ABD$  at  $D$ . Prove that  $\angle A = 90^\circ$ .

2. Let  $a, b, c$  be three distinct positive real numbers such that  $abc = 1$ . Prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

3. Let  $a, b, c, d, e, f$  be positive integers such that

$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}.$$

Suppose  $af - be = -1$ . Show that  $d \geq b + f$ .

4. There are 100 countries participating in an olympiad. Suppose  $n$  is a positive integer such that each of the 100 countries is willing to communicate in exactly  $n$  languages. If each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of  $n$ ?

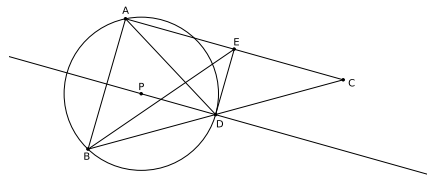
5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be the incentre of  $ABC$ . Extend  $AI$  and  $CI$ ; let them intersect  $BC$  in  $D$  and  $AB$  in  $E$  respectively. Draw a line perpendicular to  $AI$  at  $I$  to meet  $AC$  in  $J$ ; draw a line perpendicular to  $CI$  at  $I$  to meet  $AC$  in  $K$ . Suppose  $DJ = EK$ . Prove that  $BA = BC$ .

6. (a) Given any natural number  $N$ , prove that there exists a strictly increasing sequence of  $N$  positive integers in harmonic progression.

(b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

1. Let  $ABC$  be a triangle and  $D$  be the mid-point of  $BC$ . Suppose the angle bisector of  $\angle ADC$  is tangent to the circumcircle of triangle  $ABD$  at  $D$ . Prove that  $\angle A = 90^\circ$ .

**Solution:** Let  $P$  be the center of the circumcircle  $\Gamma$  of  $\triangle ABC$ . Let the tangent at  $D$  to  $\Gamma$  intersect  $AC$  in  $E$ . Then  $PD \perp DE$ . Since  $DE$  bisects  $\angle ADC$ , this implies that  $DP$  bisects  $\angle ADB$ . Hence the circumcenter and the incenter of  $\triangle ABD$  lies on the same line  $DP$ . This implies that  $DA = DB$ . Thus  $DA = DB = DC$  and hence  $D$  is the circumcenter of  $\triangle ABC$ . This gives  $\angle A = 90^\circ$ .



2. Let  $a, b, c$  be positive real numbers such that  $abc = 1$  Prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

**Solution:** Observe that

$$\begin{aligned} \frac{1}{(a-b)(a-c)} &= \frac{(b-c)}{(a-b)(b-c)(a-c)} \\ &= \frac{(a-c) - (a-b)}{(a-b)(b-c)(a-c)} = \frac{1}{(a-b)(b-c)} - \frac{1}{(b-c)(a-c)}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} &= \frac{a^3 - b^3}{(a-b)(b-c)} + \frac{c^3 - a^3}{(c-a)(c-b)} \\ &= \frac{a^2 + ab + b^2}{b-c} - \frac{c^2 + ca + a^2}{b-c} \\ &= \frac{ab + b^2 - c^2 - ca}{b-c} \\ &= \frac{(a+b+c)(b-c)}{b-c} = a + b + c. \end{aligned}$$

Therefore

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} = a + b + c \geq 3(abc)^{1/3} = 3.$$

3. Let  $a, b, c, d, e, f$  be positive integers such that

$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}.$$

Suppose  $af - be = -1$ . Show that  $d \geq b + f$ .

**Solution:** Since  $bc - ad > 0$ , we have  $bc - ad \geq 1$ . Similarly, we obtain  $de - fc \geq 1$ . Therefore

$$d = d(be - af) = dbe - daf = dbe - bfc + bfc - adf = b(de - fc) + f(bc - ad) \geq b + f.$$

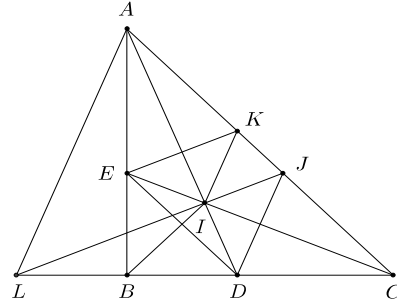
4. There are 100 countries participating in an olympiad. Suppose  $n$  is a positive integer such that each of the 100 countries is willing to communicate in exactly  $n$  languages. If each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of  $n$ ?

**Solution:** We show that  $n = 20$ . We first show that  $n = 20$  is possible. Call the countries  $C_1, \dots, C_{100}$ . Let  $1, 2, \dots, 21$  be languages and suppose, the country  $C_i (1 \leq i \leq 20)$  communicates exactly in the languages  $\{j : 1 \leq j \leq 20, j \neq i\}$ . Suppose each of the countries  $C_{21}, \dots, C_{100}$  communicates in the languages  $1, 2, \dots, 20$ . Then, clearly every set of 20 countries have a common language of communication.

Now, we show that  $n \geq 20$ . If  $n < 20$ , look at any country  $A$  communicating in the languages  $L_1, \dots, L_n$ . As no language is common to all 100 countries, for each  $L_i$ , there is a country  $A_i$  not communicating in  $L_i$ . Then, the  $n + 1 \leq 20$  countries  $A, A_1, A_2, \dots, A_n$  have no common language of communication. This contradiction shows  $n \geq 20$ .

5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $I$  be the incentre of  $ABC$ . Extend  $AI$  and  $CI$ ; let them intersect  $BC$  in  $D$  and  $AB$  in  $E$  respectively. Draw a line perpendicular to  $AI$  at  $I$  to meet  $AC$  in  $J$ ; draw a line perpendicular to  $CI$  at  $I$  to meet  $AC$  in  $K$ . Suppose  $DJ = EK$ . Prove that  $BA = BC$ .

**Solution:** Extend  $JI$  to meet  $CB$  extended at  $L$ . Then  $ALBI$  is a cyclic quadrilateral. Therefore  $\angle BLI = \angle BAI = \angle IAC$ . Therefore  $\angle LAD = \angle IBD = 45^\circ$ . Since  $\angle AIL = 90^\circ$ , it follows that  $\angle ALI = 45^\circ$ . Therefore  $AI = IL$ . This shows that the triangles  $AIJ$  and  $LID$  are congruent giving  $IJ = ID$ . Similarly,  $IK = IE$ . Since  $IJ \perp ID$  and  $IK \perp IE$  and since  $DJ = EK$ , we see that  $IJ = ID = IK = IE$ . Thus  $E, D, J, K$  are concyclic. This implies together with  $DJ = EK$  that  $EDJK$  is an isosceles trapezium. Therefore  $DE \parallel JK$ . Hence  $\angle EDA = \angle DAC = \angle A/2$  and  $\angle DEC = \angle ECA = \angle C/2$ . Since  $IE = ID$ , it follows that  $\angle A/2 = \angle C/2$ . This means  $BA = BC$ .



6. (a) Given any natural number  $N \geq 3$ , prove that there exists a strictly increasing sequence of  $N$  positive integers in harmonic progression.  
 (b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

**Solution:** (a) Let  $N \geq 3$  be a given positive integer. Consider the HP

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{N}.$$

If we multiply this by  $N!$ , we get the HP

$$N!, \frac{N!}{2}, \frac{N!}{3}, \frac{N!}{4}, \dots, \frac{N!}{N}.$$

This is decreasing. We write this in reverse order to get a strictly increasing HP:

$$\frac{N!}{N}, \frac{N!}{N-1}, \frac{N!}{N-2}, \dots, \frac{N!}{3}, \frac{N!}{2}, N!.$$

- (b) Assume the contrary that there is an infinite strictly increasing sequence  $\langle a_1, a_2, a_3, \dots \rangle$  of positive integers which forms a harmonic progression. Define  $b_k = 1/a_k$ , for  $k \geq 1$ . Then  $\langle b_1, b_2, b_3, \dots \rangle$  is a strictly decreasing sequence of positive rational numbers which is in an arithmetic progression.

Let  $d = b_2 - b_1 < 0$  be its common difference. Then  $b_1 - b_2 > 0$ . Choose a positive integer  $n$  such that

$$n > \frac{b_1}{b_1 - b_2}.$$

Then

$$b_{n+1} = b_1 + (b_2 - b_1)n = b_1 - (b_1 - b_2)n < b_1 - \left(\frac{b_1}{b_1 - b_2}\right) \times (b_1 - b_2) = 0.$$

Thus for all large  $n$ , we see that  $b_n$  is negative contradicting  $b_n$  is positive for all  $n$ .

—————0—————

# Regional Mathematical Olympiad-2016

Time: 3 hours

October 23, 2016

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be an isosceles triangle with  $AB = AC$ . Let  $\Gamma$  be its circumcircle and let  $O$  be the centre of  $\Gamma$ . Let  $CO$  meet  $\Gamma$  in  $D$ . Draw a line parallel to  $AC$  through  $D$ . Let it intersect  $AB$  at  $E$ . Suppose  $AE : EB = 2 : 1$ . Prove that  $ABC$  is an equilateral triangle.

2. Let  $a, b, c$  be positive real numbers such that

$$\frac{ab}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1.$$

Prove that  $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}$ .

3. The present ages in years of two brothers  $A$  and  $B$ , and their father  $C$  are three distinct positive integers  $a, b$ , and  $c$  respectively. Suppose  $\frac{b-1}{a-1}$  and  $\frac{b+1}{a+1}$  are two consecutive integers, and  $\frac{c-1}{b-1}$  and  $\frac{c+1}{b+1}$  are two consecutive integers. If  $a+b+c \leq 150$  determine  $a, b$  and  $c$ .

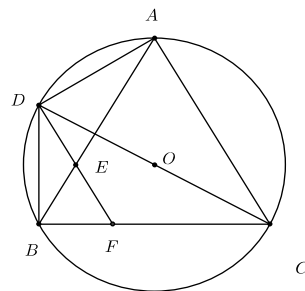
4. A box contains 4032 answer scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

5. Let  $ABC$  be a triangle,  $AD$  an altitude and  $AE$  a median. Assume  $B, D, E, C$  lie in that order on the line  $BC$ . Suppose the incentre of triangle  $ABE$  lies on  $AD$  and the incentre of  $ADC$  lies on  $AE$ . Find, with proof, the angles of triangle  $ABC$ .

6. (i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.
- (ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

1. Let  $ABC$  be an isosceles triangle with  $AB = AC$ . Let  $\Gamma$  be its circumcircle and let  $O$  be the centre of  $\Gamma$ . Let  $CO$  meet  $\Gamma$  in  $D$ . Draw a line parallel to  $AC$  through  $D$ . Let it intersect  $AB$  at  $E$ . Suppose  $AE : EB = 2 : 1$ . Prove that  $ABC$  is an equilateral triangle.

**Solution:** Extend  $DE$  to meet  $BC$  at  $F$ . Join  $BD$  and  $DA$ . Since  $CD$  is a diameter, we see that  $\angle DBC = 90^\circ$ . Since  $DF$  is parallel to  $AC$ , it follows that  $\triangle EBF \sim \triangle ABC$ . Hence  $EB = EF$ . Now  $\angle EBF = \angle EFB = 90^\circ - \angle EDB$ . But  $\angle EBF + \angle EBD = 90^\circ$ . Hence we obtain  $\angle EBD = \angle EDB$ , which gives  $EB = ED$ . Since  $AE : EB = 2 : 1$  and  $EB = ED$ , we obtain  $AE = 2ED$ . Hence  $\angle DAB = 30^\circ$ . This implies  $\angle DCB = 30^\circ$  and hence  $\angle BDC = 60^\circ$ . But then  $\angle BAC = \angle BDC = 60^\circ$  and hence  $\triangle ABC$  is equilateral



2. Let  $a, b, c$  be positive real numbers such that

$$\frac{ab}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1.$$

Prove that  $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}$ .

**Solution:** The given condition is equivalent to

$$\sum ab(1+ca)(1+ab) = (1+ab)(1+bc)(1+ca).$$

This gives

$$\sum ab + \sum a^2b^2 + abc \sum a + abc \sum a^2b = 1 + \sum ab + abc \sum a + a^2b^2c^2.$$

Hence

$$a^2b^2c^2 + 1 = \sum a^2b^2 + abc \sum a^2b.$$

Using

$$\sum a^2b^2 \geq 3(abc)^{4/3}, \quad \sum a^2b \geq 3abc,$$

we get

$$a^2b^2c^2 + 1 \geq 3(abc)^{4/3} + 3(abc)^2.$$

Taking  $x = (abc)^{2/3}$ , this reduces to  $2x^3 + 3x^2 - 1 \leq 0$ . This gives  $(x+1)^2(2x-1) \leq 0$ . Hence  $x \leq 1/2$ . Therefore  $abc \leq 1/2\sqrt{2}$ . Finally

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{3}{abc} \geq 6\sqrt{2}.$$

3. The present ages in years of two brothers  $A$  and  $B$ , and their father  $C$  are three distinct positive integers  $a, b$ , and  $c$  respectively. Suppose  $\frac{b-1}{a-1}$  and  $\frac{b+1}{a+1}$  are two consecutive integers, and  $\frac{c-1}{b-1}$  and  $\frac{c+1}{b+1}$  are two consecutive integers. If  $a+b+c \leq 150$  determine  $a, b$  and  $c$ .

**Solution:** We have

$$\frac{b-1}{a-1} = l, \quad \frac{b+1}{a+1} = l-1, \quad \frac{c-1}{b-1} = m, \quad \frac{c+1}{b+1} = m-1$$

(If we take  $a \leq b$ , we see that  $\frac{b-1}{a-1} \geq \frac{b+1}{a+1}$ .)

Consider the first two relations:  $b - 1 = l(a - 1)$ ,  $b + 1 = (l - 1)(a + 1)$ . Solving for  $a$ , we get  $a = 2l - 3$  and hence  $b = 2l^2 - 4l + 1$ . Using the second set of relations, we obtain  $b = 2m - 3$  and  $c = 2m^2 - 4m + 1$ . Thus we have  $2m - 3 = 2l^2 - 4l + 1$  or  $m = (l - 1)^2 + 1$ . Obviously  $l > 1$ . If  $l = 2$ , we get  $a = 1$  which forces  $b = 1$  and  $c = 1$ , which is impossible. If  $l = 3$ , we get  $a = 3$ ,  $b = 7$  and  $c = 31$ . If  $l \geq 4$ , then  $m \geq 10$  and  $c \geq 161$ . But then  $a + b + c > 150$ . Thus the only choice is  $a = 3$ ,  $b = 7$  and  $c = 31$ .

4. A box contains answer 4032 scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

**Solution:** There are three types of processes. In the first type, the scripts with odd scores decreases by 2. In the second and third types, there is no change in the number of scripts with odd scores. Hence at each step, the number of scripts with odd score decreases by 0 or 2. Since there are 2016 scripts with odd scores, the number of scripts with odd scores at the end is either 0 or 2. Since it is given that there is at least one script with odd scores, two of the three must have odd scores.

5. Let  $ABC$  be a triangle,  $AD$  an altitude and  $AE$  a median. Assume  $B, D, E, C$  lie in that order on the line  $BC$ . Suppose the incentre of triangle  $ABE$  lies on  $AD$  and the incentre of  $ADC$  lies on  $AE$ . Find the angles of triangle  $ABC$ .

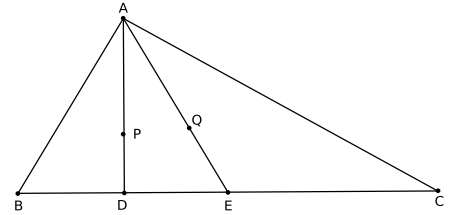
**Solution:** Since  $AD \perp BE$  and the incentre of  $\triangle ABE$  lies on  $AD$ , it follows that  $ABE$  is isosceles. In particular  $\angle BAD = \angle DAE = \alpha$ , say. Since  $AE$  is the bisector of  $\angle DAC$ , it follows that  $\angle EAC = \angle DAE = \alpha$ . Moreover, we have

$$\frac{AD}{AC} = \frac{DE}{CE}.$$

Since  $BE = EC = \frac{a}{2}$ , we also have  $DE = \frac{1}{2}BE = \frac{a}{4}$ . Thus we get

$$\frac{AD}{AC} = \frac{DE}{EC} = \frac{a/4}{a/2} = \frac{1}{2}.$$

Since  $\triangle ADE$  is a right-angled triangle and  $AD = AC/2$ , it follows that  $\angle ACD = 30^\circ$ . Hence  $\angle DAC = 60^\circ$ . Since  $\angle DAC = 2\alpha$ , we get  $\alpha = 30^\circ$ . Now  $\angle A = 3\alpha$  and hence  $\angle A = 90^\circ$ . This gives  $\angle B = 60^\circ$ .



6. i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.  
(ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

**Solution:**

i) Let  $a$  be the first term of the AP and  $d$  be the common difference. (Here  $a, d$  are positive integers.) We can find an integer  $b$  such that  $b^3 = a + (n - 1)d$  for some  $n \in \mathbb{N}$ . Consider  $(b + d)^3$ . We observe that

$$(b + d)^3 = b^3 + d(3b^2 + 3bd + d^2) = a + (n - 1 + 3b^2 + 3bd + d^2)d = a + (m - 1)d,$$

where  $m = n + 3b^2 + 3bd + d^2$  is a positive integer. Hence  $(b + d)^3$  is also in the given AP. More generally, the same method shows that  $(b + kd)^3$  is in the AP for every  $k \in \mathbb{N}$ . Hence the given AP contains infinitely many cubes.

ii) Consider the AP  $\langle 2, 6, 10, 14, \dots \rangle$ . Here  $a = 2$  and  $d = 4$ . The general term is  $2 + 4k$ , where  $k \geq 0$  is an integer. Suppose  $2 + 4k = b^3$  for some integer  $b$ . Then 2 divides  $b$ . Hence  $b = 2c$  for some  $c$ . Therefore  $8c^3 = 2 + 4k$  or  $4c^3 = 2k + 1$ . But this is impossible since LHS is even and RHS is odd. We conclude that the AP  $\langle 2, 6, 10, 14, \dots \rangle$  does not contain any cube.

—————0—————

**Regional Mathematical Olympiad-2000**  
**Problems and Solutions**

1. Let  $AC$  be a line segment in the plane and  $B$  a point between  $A$  and  $C$ . Construct isosceles triangles  $PAB$  and  $QBC$  on one side of the segment  $AC$  such that  $\angle APB = \angle BQC = 120^\circ$  and an isosceles triangle  $RAC$  on the otherside of  $AC$  such that  $\angle ARC = 120^\circ$ . Show that  $PQR$  is an equilateral triangle.

**Solution:** We give here 2 different solutions.

1. Drop perpendiculars from  $P$  and  $Q$  to  $AC$  and extend them to meet  $AR, RC$  in  $K, L$  respectively. Join  $KB, PB, QB, LB, KL$ . (Fig.1.)

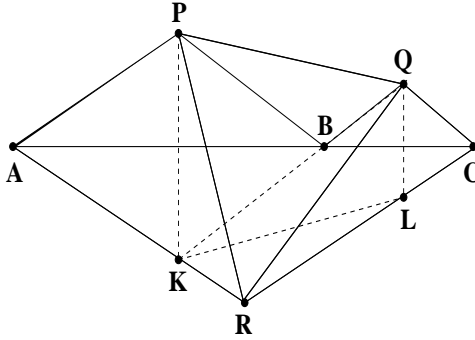


Fig. 1

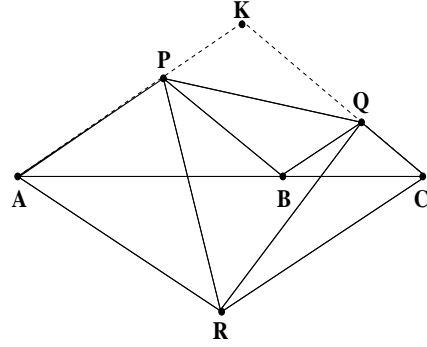


Fig. 2

Observe that  $K, B, Q$  are collinear and so are  $P, B, L$ . ( This is because  $\angle QBC = \angle PBA = \angle KBA$  and similarly  $\angle PBA = \angle CBL$ .) By symmetry we see that  $\angle KPQ = \angle PKL$  and  $\angle KPB = \angle PKB$ . It follows that  $\angle LPQ = \angle LKQ$  and hence  $K, L, Q, P$  are concyclic. We also note that  $\angle KPL + \angle KRL = 60^\circ + 120^\circ = 180^\circ$ . This implies that  $P, K, R, L$  are concyclic. We conclude that  $P, K, R, L, Q$  are concyclic. This gives

$$\angle PRQ = \angle PKQ = 60^\circ, \quad \angle RPQ = \angle RKQ = \angle RAP = 60^\circ.$$

2. Produce  $AP$  and  $CQ$  to meet at  $K$ . Observe that  $AKCR$  is a rhombus and  $BQKP$  is a parallelogram. (See Fig.2.) Put  $AP = x, CQ = y$ . Then  $PK = BQ = y$ ,  $KQ = PB = x$  and  $AR = RC = CK = KA = x + y$ . Using cosine rule in triangle  $PKQ$ , we get  $PQ^2 = x^2 + y^2 - 2xy \cos 120^\circ = x^2 + y^2 + xy$ . Similarly cosine rule in triangle  $QCR$  gives  $QR^2 = y^2 + (x + y)^2 - 2xy \cos 60^\circ = x^2 + y^2 + xy$  and cosine rule in triangle  $PAR$  gives  $RP^2 = x^2 + (x + y)^2 - 2xy \cos 60^\circ = x^2 + y^2 + xy$ . It follows that  $PQ = QR = RP$ .

2. Solve the equation  $y^3 = x^3 + 8x^2 - 6x + 8$ , for positive integers  $x$  and  $y$ .

**Solution:** We have

$$y^3 - (x+1)^3 = x^3 + 8x^2 - 6x + 8 - (x^3 + 3x^2 + 3x + 1) = 5x^2 - 9x + 7.$$

Consider the quadratic equation  $5x^2 - 9x + 7 = 0$ . The discriminant of this equation is  $D = 9^2 - 4 \times 5 \times 7 = -59 < 0$  and hence the expression  $5x^2 - 9x + 7$  is positive for all real values of  $x$ . We conclude that  $(x+1)^3 < y^3$  and hence  $x+1 < y$ .

On the other hand we have

$$(x+3)^3 - y^3 = x^3 + 9x^2 + 27x + 27 - (x^3 + 8x^2 - 6x + 8) = x^2 + 33x + 19 > 0$$

for all positive  $x$ . We conclude that  $y < x+3$ . Thus we must have  $y = x+2$ . Putting this value of  $y$ , we get

$$0 = y^3 - (x+2)^3 = x^3 + 8x^2 - 6x + 8 - (x^3 + 6x^2 + 12x + 8) = 2x^2 - 18x.$$

We conclude that  $x = 0$  and  $y = 2$  or  $x = 9$  and  $y = 11$ .

3. Suppose  $\langle x_1, x_2, \dots, x_n, \dots \rangle$  is a sequence of positive real numbers such that  $x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq \dots$ , and for all  $n$

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \dots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all  $k$  the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \dots + \frac{x_k}{k} \leq 3.$$

**Solution:** Let  $k$  be a natural number and  $n$  be the unique integer such that  $(n-1)^2 \leq k < n^2$ . Then we see that

$$\begin{aligned} \sum_{r=1}^k \frac{x_r}{r} &\leq \left( \frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} \right) + \left( \frac{x_4}{4} + \frac{x_5}{5} + \dots + \frac{x_8}{8} \right) \\ &\quad + \dots + \left( \frac{x_{(n-1)^2}}{(n-1)^2} + \dots + \frac{x_k}{k} + \dots + \frac{x_{n^2-1}}{n^2-1} \right) \\ &\leq \left( \frac{x_1}{1} + \frac{x_1}{1} + \frac{x_1}{1} \right) + \left( \frac{x_4}{4} + \frac{x_4}{4} + \dots + \frac{x_4}{4} \right) \\ &\quad + \dots + \left( \frac{x_{(n-1)^2}}{(n-1)^2} + \dots + \frac{x_{(n-1)^2}}{(n-1)^2} \right) \\ &= \frac{3x_1}{1} + \frac{5x_2}{4} + \dots + \frac{(2n-1)x_{(n-1)^2}}{(n-1)^2} \\ &= \sum_{r=1}^{n-1} \frac{(2r+1)x_{r^2}}{r^2} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{r=1}^{n-1} \frac{3r}{r^2} x_{r^2} \\
&= 3 \sum_{r=1}^{n-1} \frac{x_{r^2}}{r} \leq 3,
\end{aligned}$$

where the last inequality follows from the given hypothesis.

4. All the 7-digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once, and not divisible by 5, are arranged in the increasing order. Find the 2000-th number in this list.

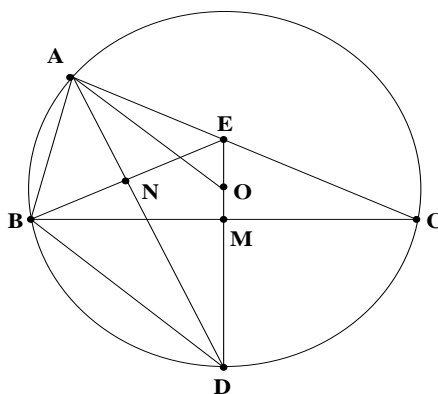
**Solution:** The number of 7-digit numbers with 1 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once is  $6! = 720$ . But 120 of these end in 5 and hence are divisible by 5. Thus the number of 7-digit numbers with 1 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once but not divisible by 5 is 600. Similarly the number of 7-digit numbers with 2 and 3 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once but not divisible by 5 is also 600 each. These account for 1800 numbers. Hence 2000-th number must have 4 in the left most place.

Again the number of such 7-digit numbers beginning with 41, 42 and not divisible by 5 is  $120 - 24 = 96$  each and these account for 192 numbers. This shows that 2000-th number in the list must begin with 43.

The next 8 numbers in the list are: 4312567, 4312576, 4312657, 4312756, 4315267, 4315276, 4315627 and 4315672. Thus 2000-th number in the list is 4315672.

5. The internal bisector of angle  $A$  in a triangle  $ABC$  with  $AC > AB$ , meets the circumcircle  $\Gamma$  of the triangle in  $D$ . Join  $D$  to the centre  $O$  of the circle  $\Gamma$  and suppose  $DO$  meets  $AC$  in  $E$ , possibly when extended. Given that  $BE$  is perpendicular to  $AD$ , show that  $AO$  is parallel to  $BD$ .

**Solution:** We consider here the case when  $ABC$  is an acute-angled triangle; the cases when  $\angle A$  is obtuse or one of the angles  $\angle B$  and  $\angle C$  is obtuse may be handled similarly.



Let  $M$  be the point of intersection of  $DE$  and  $BC$ ; let  $AD$  intersect  $BE$  in  $N$ . Since  $ME$  is the perpendicular bisector of  $BC$ , we have  $BE = CE$ . Since  $AN$  is the internal bisector of  $\angle A$ , and is perpendicular to  $BE$ , it must bisect  $BE$ ; i.e.,  $BN = NE$ . This in turn implies that  $DN$  bisects  $\angle BDE$ . But  $\angle BDA = \angle BCA = \angle C$ . Thus  $\angle ODA = \angle C$ . Since  $OD = OA$ , we get  $\angle OAD = \angle C$ . It follows that  $\angle BDA = \angle C = \angle OAD$ . This implies that  $OA$  is parallel to  $BD$ .

6. (i) Consider two positive integers  $a$  and  $b$  which are such that  $a^a b^b$  is divisible by 2000. What is the least possible value of the product  $ab$ ?  
(ii) Consider two positive integers  $a$  and  $b$  which are such that  $a^b b^a$  is divisible by 2000. What is the least possible value of the product  $ab$ ?

**Solution:** We have  $2000 = 2^4 5^3$ .

(i) Since 2000 divides  $a^a b^b$ , it follows that 2 divides  $a$  or  $b$  and similarly 5 divides  $a$  or  $b$ . In any case 10 divides  $ab$ . Thus the least possible value of  $ab$  for which  $2000 | a^a b^b$  must be a multiple of 10. Since 2000 divides  $10^{10} 1^1$ , we can take  $a = 10, b = 1$  to get the least value of  $ab$  equal to 10.

(ii) As in (i) we conclude that 10 divides  $ab$ . Thus the least value of  $ab$  for which  $2000 | a^b b^a$  is again a multiple of 10. If  $ab = 10$ , then the possibilities are  $(a, b) = (1, 10), (2, 5), (5, 2), (10, 1)$ . But in all these cases it is easy to verify that 2000 does not divide  $a^b b^a$ . The next multiple of 10 is 20. In this case we can take  $(a, b) = (4, 5)$  and verify that 2000 divides  $4^5 5^4$ . Thus the least value here is 20.

7. Find all real values of  $a$  for which the equation  $x^4 - 2ax^2 + x + a^2 - a = 0$  has all its roots real.

**Solution:** Let us consider  $x^4 - 2ax^2 + x + a^2 - a = 0$  as a quadratic equation in  $a$ . We see that the roots are

$$a = x^2 + x, \quad a = x^2 - x + 1.$$

Thus we get a factorisation

$$(a - x^2 - x)(a - x^2 + x - 1) = 0.$$

It follows that  $x^2 + x = a$  or  $x^2 - x + 1 = a$ . Solving these we get

$$x = \frac{-1 \pm \sqrt{1 + 4a}}{2}, \quad \text{or} \quad x = \frac{-1 \pm \sqrt{4a - 3}}{2}.$$

Thus all the four roots are real if and only if  $a \geq 3/4$ .

# Regional Mathematical Olympiad – 2001

Time: 3 hours

2 December 2001

1. Let  $BE$  and  $CF$  be the altitudes of an acute triangle  $ABC$ , with  $E$  on  $AC$  and  $F$  on  $AB$ . Let  $O$  be the point of intersection of  $BE$  and  $CF$ . Take any line  $KL$  through  $O$  with  $K$  on  $AB$  and  $L$  on  $AC$ . Suppose  $M$  and  $N$  are located on  $BE$  and  $CF$  respectively, such that  $KM$  is perpendicular to  $BE$  and  $LN$  is perpendicular to  $CF$ . Prove that  $FM$  is parallel to  $EN$ .
2. Find all primes  $p$  and  $q$  such that  $p^2 + 7pq + q^2$  is the square of an integer.
3. Find the number of positive integers  $x$  which satisfy the condition

$$\left[ \frac{x}{99} \right] = \left[ \frac{x}{101} \right].$$

(Here  $[z]$  denotes, for any real  $z$ , the largest integer not exceeding  $z$ ; e.g.  $[7/4] = 1$ .)

4. Consider an  $n \times n$  array of numbers:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Suppose each row consists of the  $n$  numbers  $1, 2, 3, \dots, n$  in some order and  $a_{ij} = a_{ji}$  for  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, n$ . If  $n$  is odd, prove that the numbers  $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$  are  $1, 2, 3, \dots, n$  in some order.

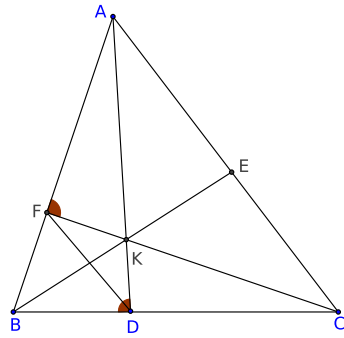
5. In a triangle  $ABC$ ,  $D$  is a point on  $BC$  such that  $AD$  is the internal bisector of  $\angle A$ . Suppose  $\angle B = 2\angle C$  and  $CD = AB$ . Prove that  $\angle A = 72^\circ$ .
6. If  $x, y, z$  are the sides of a triangle. then prove that

$$|x^2(y - z) + y^2(z - x) + z^2(x - y)| < xyz.$$

7. Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 positive odd integers by a multiple of 2001.

## Problems and Solutions: CRMO-2011

1. Let  $ABC$  be a triangle. Let  $D, E, F$  be points respectively on the segments  $BC, CA, AB$  such that  $AD, BE, CF$  concur at the point  $K$ . Suppose  $BD/DC = BF/FA$  and  $\angle ADB = \angle AFC$ . Prove that  $\angle ABE = \angle CAD$ .



**Solution:** Since  $BD/DC = BF/FA$ , the lines  $DF$  and  $CA$  are parallel. We also have  $\angle BDK = \angle ADB = \angle AFC = 180^\circ - \angle BFK$ , so that  $BDKF$  is a cyclic quadrilateral. Hence  $\angle FBK = \angle FDK$ . Finally, we get

$$\begin{aligned}\angle ABE &= \angle FBK = \angle FDK \\ &= \angle FDA = \angle DAC,\end{aligned}$$

since  $FD \parallel AC$ .

2. Let  $(a_1, a_2, a_3, \dots, a_{2011})$  be a permutation (that is a rearrangement) of the numbers  $1, 2, 3, \dots, 2011$ . Show that there exist two numbers  $j, k$  such that  $1 \leq j < k \leq 2011$  and  $|a_j - j| = |a_k - k|$ .

**Solution:** Observe that  $\sum_{j=1}^{2011} (a_j - j) = 0$ , since  $(a_1, a_2, a_3, \dots, a_{2011})$  is a permutation of  $1, 2, 3, \dots, 2011$ . Hence  $\sum_{j=1}^{2011} |a_j - j|$  is even. Suppose  $|a_j - j| \neq |a_k - k|$  for all  $j \neq k$ . This means the collection  $\{|a_j - j| : 1 \leq j \leq 2011\}$  is the same as the collection  $\{0, 1, 2, \dots, 2010\}$  as the maximum difference is  $2011-1=2010$ . Hence

$$\sum_{j=1}^{2011} |a_j - j| = 1 + 2 + 3 + \dots + 2010 = \frac{2010 \times 2011}{2} = 2011 \times 1005,$$

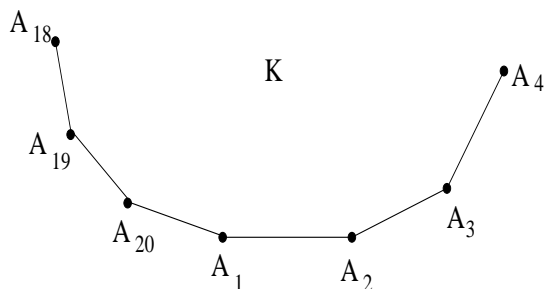
which is odd. This shows that  $|a_j - j| = |a_k - k|$  for some  $j \neq k$ .

3. A natural number  $n$  is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting  $k$  from  $n$  and the larger one is obtained by adding  $l$  to  $n$ . Prove that  $n - kl$  is a perfect square.

**Solution:** Let  $u$  be a natural number such that  $u^2 < n < (u+1)^2$ . Then  $n - k = u^2$  and  $n + l = (u+1)^2$ . Thus

$$\begin{aligned}n - kl &= n - (n - u^2)((u+1)^2 - n) \\ &= n - n(u+1)^2 + n^2 + u^2(u+1)^2 - nu^2 \\ &= n^2 + n(1 - (u+1)^2 - u^2) + u^2(u+1)^2 \\ &= n^2 + n(1 - 2u^2 - 2u - 1) + u^2(u+1)^2 \\ &= n^2 - 2nu(u+1) + (u(u+1))^2 \\ &= (n - u(u+1))^2.\end{aligned}$$

4. Consider a 20-sided convex polygon  $K$ , with vertices  $A_1, A_2, \dots, A_{20}$  in that order. Find the number of ways in which three sides of  $K$  can be chosen so that every pair among them has at least two sides of  $K$  between them. (For example  $(A_1A_2, A_4A_5, A_{11}A_{12})$  is an admissible triple while  $(A_1A_2, A_4A_5, A_{19}A_{20})$  is not.)



**Solution:** First let us count all the admissible triples having  $A_1A_2$  as one of the sides. Having chosen  $A_1A_2$ , we cannot choose  $A_2A_3$ ,  $A_3A_4$ ,  $A_{20}A_1$  nor  $A_{19}A_{20}$ . Thus we have to choose two sides separated by 2 sides among 15 sides  $A_4A_5, A_5A_6, \dots, A_{18}A_{19}$ . If  $A_4A_5$  is one of them, the choice for the remaining side is only from 12 sides

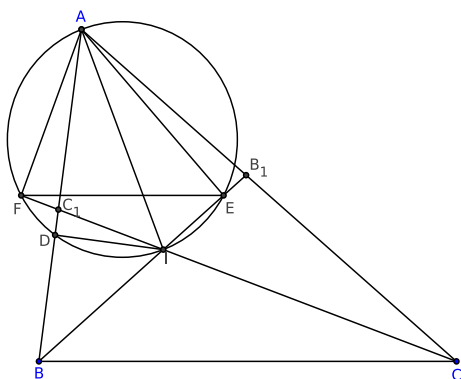
$A_7A_8, A_8A_9, \dots, A_{18}A_{19}$ . If we choose  $A_5A_6$  after  $A_1A_2$ , the choice for the third side is now only from  $A_8A_9, A_9A_{10}, \dots, A_{18}A_{19}$  (11 sides). Thus the number of choices progressively decreases and finally for the side  $A_{15}A_{16}$  there is only one choice, namely,  $A_{18}A_{19}$ . Hence the number of triples with  $A_1A_2$  as one of the sides is

$$12 + 11 + 10 + \dots + 1 = \frac{12 \times 13}{2} = 78.$$

Hence the number of triples then would be  $(78 \times 20)/3 = 520$ .

**Remark:** For an  $n$ -sided polygon, the number of such triples is  $\frac{n(n-7)(n-8)}{6}$ , for  $n \geq 9$ . We may check that for  $n = 20$ , this gives  $(20 \times 13 \times 12)/6 = 520$ .

5. Let  $ABC$  be a triangle and let  $BB_1, CC_1$  be respectively the bisectors of  $\angle B, \angle C$  with  $B_1$  on  $AC$  and  $C_1$  on  $AB$ . Let  $E, F$  be the feet of perpendiculars drawn from  $A$  onto  $BB_1, CC_1$  respectively. Suppose  $D$  is the point at which the incircle of  $ABC$  touches  $AB$ . Prove that  $AD = EF$ .



**Solution:** Observe that  $\angle ADI = \angle AFI = \angle AEI = 90^\circ$ . Hence  $A, F, D, I, E$  all lie on the circle with  $AI$  as diameter. We also know

$$\angle BIC = 90^\circ + \frac{\angle A}{2} = \angle FIE.$$

This gives

$$\begin{aligned} \angle FAE &= 180^\circ - \left(90^\circ + \frac{\angle A}{2}\right) \\ &= 90^\circ - \frac{\angle A}{2}. \end{aligned}$$

We also have  $\angle AID = 90^\circ - \frac{\angle A}{2}$ . Thus  $\angle FAE = \angle AID$ . This shows the chords  $FE$  and  $AD$  subtend equal angles at the circumference of the same circle. Hence they have equal lengths, i.e.,  $FE = AD$ .

6. Find all pairs  $(x, y)$  of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

**Solution:** Observe that

$$x^2 + y + x + y^2 + \frac{1}{2} = \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \geq 0.$$

This shows that  $x^2 + y + x + y^2 \geq (-1/2)$ . Hence we have

$$\begin{aligned} 1 = 16^{x^2+y} + 16^{x+y^2} &\geq 2 \left(16^{x^2+y} \cdot 16^{x+y^2}\right)^{1/2}, \quad (\text{by AM-GM inequality}) \\ &= 2 \left(16^{x^2+y+x+y^2}\right)^{1/2} \\ &\geq 2(16)^{-1/4} = 1. \end{aligned}$$

Thus equality holds every where. We conclude that

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 0.$$

This shows that  $(x, y) = (-1/2, -1/2)$  is the only solution, as can easily be verified.

————00000————

## Problems and Solutions... CRMO-2002

1. In an acute triangle  $ABC$ , points  $D, E, F$  are located on the sides  $BC, CA, AB$  respectively such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \quad \frac{AE}{AF} = \frac{AB}{AC}, \quad \frac{BF}{BD} = \frac{BC}{BA}.$$

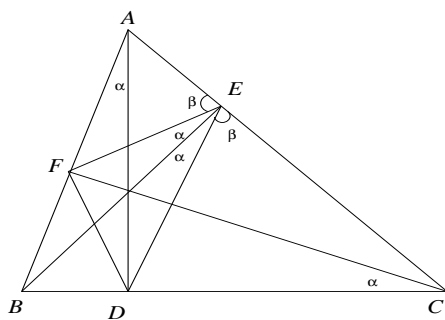
Prove that  $AD, BE, CF$  are the altitudes of  $ABC$ .

**Solution:** Put  $CD = x$ . Then with usual notations we get

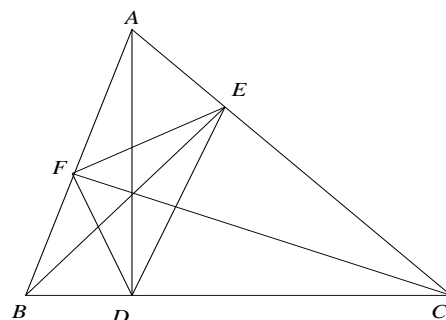
$$CE = \frac{CD \cdot CB}{CA} = \frac{ax}{b}.$$

Since  $AE = AC - CE = b - CE$ , we obtain

$$AE = \frac{b^2 - ax}{b}, \quad AF = \frac{AE \cdot AC}{AB} = \frac{b^2 - ax}{c}.$$



**Fig. 1**



**Fig. 2**

This in turn gives

$$BF = AB - AF = \frac{c^2 - b^2 + ax}{c}.$$

Finally we obtain

$$BD = \frac{c^2 - b^2 + ax}{a}.$$

Using  $BD = a - x$ , we get

$$x = \frac{a^2 - c^2 + b^2}{2a}.$$

However, if  $L$  is the foot of perpendicular from  $A$  on to  $BC$  then, using Pythagoras theorem in triangles  $ALB$  and  $ALC$  we get

$$b^2 - LC^2 = c^2 - (a - LC)^2$$

which reduces to  $LC = (a^2 - c^2 + b^2)/2a$ . We conclude that  $LC = DC$  proving  $L = D$ . Or, we can also infer that  $x = b \cos C$  from cosine rule in triangle  $ABC$ . This implies that  $CD = CL$ , since  $CL = b \cos C$  from right triangle  $ALC$ . Thus  $AD$  is altitude on to  $BC$ . Similar proof works for the remaining altitudes.

Alternately, we see that  $CD \cdot CB = CE \cdot CA$ , so that  $ABDE$  is a cyclic quadrilateral. Similarly we infer that  $BCEF$  and  $CAFD$  are also cyclic quadrilaterals. (See Fig. 2.) Thus  $\angle AEF = \angle B = \angle CED$ . Moreover  $\angle BED = \angle DAF = \angle DCF = \angle BCF = \angle BEF$ . It follows that  $\angle BEA = \angle BEC$  and hence each is a right angle thus proving that  $BE$  is an altitude. Similarly we prove that  $CF$  and  $AD$  are altitudes. (Note that the concurrence of the lines  $AD$ ,  $BE$ ,  $CF$  are not required.)

2. Solve the following equation for real  $x$ :

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

**Solution:** By setting  $u = x^2 + x - 2$  and  $v = 2x^2 - x - 1$ , we observe that the equation reduces to  $u^3 + v^3 = (u + v)^3$ . Since  $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$ , it follows that  $uv(u + v) = 0$ . Hence  $u = 0$  or  $v = 0$  or  $u + v = 0$ . Thus we obtain  $x^2 + x - 2 = 0$  or  $2x^2 - x - 1 = 0$  or  $x^2 - 1 = 0$ . Solving each of them we get  $x = 1, -2$  or  $x = 1, -1/2$  or  $x = 1, -1$ . Thus  $x = 1$  is a root of multiplicity 3 and the other roots are  $-1, -2, -1/2$ .

(Alternately, it can be seen that  $x - 1$  is a factor of  $x^2 + x - 2$ ,  $2x^2 - x - 1$  and  $x^2 - 1$ . Thus we can write the equation in the form

$$(x - 1)^3(x + 2)^3 + (x - 1)^3(2x + 1)^3 = 27(x - 1)^3(x + 1)^3.$$

Thus it is sufficient to solve the cubic equation

$$(x + 2)^3 + (2x + 1)^3 = 27(x + 1)^3.$$

This can be solved as earlier or expanding every thing and simplifying the relation.)

3. Let  $a, b, c$  be positive integers such that  $a$  divides  $b^2$ ,  $b$  divides  $c^2$  and  $c$  divides  $a^2$ . Prove that  $abc$  divides  $(a + b + c)^7$ .

**Solution:** Consider the expansion of  $(a + b + c)^7$ . We show that each term here is divisible by  $abc$ . It contains terms of the form  $r_{klm}a^k b^l c^m$ , where  $r_{klm}$  is a constant (some binomial coefficient) and  $k, l, m$  are nonnegative integers such that  $k + l + m = 7$ . If  $k \geq 1, l \geq 1, m \geq 1$ , then  $abc$  divides  $a^k b^l c^m$ . Hence we have to consider terms in which one or two of  $k, l, m$  are zero. Suppose for example  $k = l = 0$  and consider  $c^7$ . Since  $b$  divides  $c^2$  and  $a$  divides  $c^4$ , it follows that  $abc$  divides  $c^7$ . A similar argument gives the result for  $a^7$  or  $b^7$ . Consider the case in which two indices are nonzero, say for example,  $bc^6$ . Since  $a$  divides  $c^4$ , here again  $abc$  divides  $bc^6$ . If we take  $b^2c^5$ , then also using  $a$  divides  $c^4$  we obtain the result. For  $b^3c^4$ , we use the fact that  $a$  divides  $b^2$ . Similar argument works for  $b^4c^3$ ,  $b^5c^2$  and  $b^6c$ . Thus each of the terms in the expansion of  $(a + b + c)^7$  is divisible by  $abc$ .

4. Suppose the integers  $1, 2, 3, \dots, 10$  are split into two disjoint collections  $a_1, a_2, a_3, a_4, a_5$  and  $b_1, b_2, b_3, b_4, b_5$  such that

$$a_1 < a_2 < a_3 < a_4 < a_5,$$

$$b_1 > b_2 > b_3 > b_4 > b_5.$$

- (i) Show that the larger number in any pair  $\{a_j, b_j\}$ ,  $1 \leq j \leq 5$ , is at least 6.  
(ii) Show that  $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$  for every such partition.

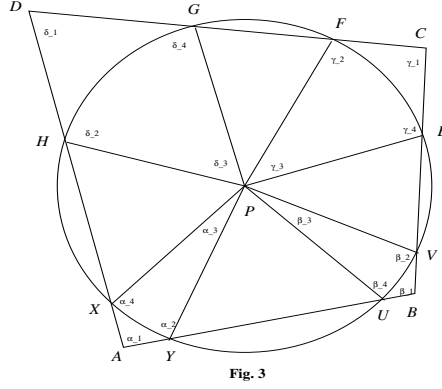
**Solution:**

- (i) Fix any pair  $\{a_j, b_j\}$ . We have  $a_1 < a_2 < \dots < a_{j-1} < a_j$  and  $b_j > b_{j+1} > \dots > b_5$ . Thus there are  $j-1$  numbers smaller than  $a_j$  and  $5-j$  numbers smaller than  $b_j$ . Together they account for  $j-1+5-j=4$  distinct numbers smaller than  $a_j$  as well as  $b_j$ . Hence the larger of  $a_j$  and  $b_j$  is at least 6.
- (ii) The first part shows that the larger numbers in the pairs  $\{a_j, b_j\}$ ,  $1 \leq j \leq 5$ , are 6, 7, 8, 9, 10 and the smaller numbers are 1, 2, 3, 4, 5. This implies that

$$\begin{aligned} |a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| \\ = 10 + 9 + 8 + 7 + 6 - (1 + 2 + 3 + 4 + 5) = 25. \end{aligned}$$

5. The circumference of a circle is divided into eight arcs by a convex quadrilateral  $ABCD$ , with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by  $p, q, r, s$  in counter-clockwise direction starting from some arc. Suppose  $p + r = q + s$ . Prove that  $ABCD$  is a cyclic quadrilateral.

**Solution:** Let the lengths of the arcs  $XY, UV, EF, GH$  be respectively  $p, q, r, s$ . We also use the following notations: (See figure)



$\angle XAY = \alpha_1, \angle AYP = \alpha_2, \angle YPX = \alpha_3, \angle PXA = \alpha_4, \angle UBY = \beta_1, \angle BVP = \beta_2, \angle VPU = \beta_3, \angle PUB = \beta_4, \angle ECF = \gamma_1, \angle CFP = \gamma_2, \angle FPE = \gamma_3, \angle PEC = \gamma_4, \angle GDH = \delta_1, \angle DHP = \delta_2, \angle HPG = \delta_3, \angle PGD = \delta_4$ .

We observe that

$$\sum \alpha_j = \sum \beta_j = \sum \gamma_j = \sum \delta_j = 2\pi.$$

It follows that

$$\sum (\alpha_j + \gamma_j) = \sum (\beta_j + \delta_j).$$

On the other hand, we also have  $\alpha_2 = \beta_4$  since  $PY = PU$ . Similarly we have other relations:  $\beta_2 = \gamma_4, \gamma_2 = \delta_4$  and  $\delta_2 = \alpha_4$ . It follows that

$$\alpha_1 + \alpha_3 + \gamma_1 + \gamma_3 = \beta_1 + \beta_3 + \delta_1 + \delta_3.$$

But  $p + r = q + s$  implies that  $\alpha_3 + \gamma_3 = \beta_3 + \delta_3$ . We thus obtain

$$\alpha_1 + \gamma_1 = \beta_1 + \delta_1.$$

Since  $\alpha_1 + \gamma_1 + \beta_1 + \delta_1 = 360^\circ$ , it follows that  $ABCD$  is a cyclic quadrilateral.

6. For any natural number  $n > 1$ , prove the inequality:

$$\frac{1}{2} < \frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} < \frac{1}{2} + \frac{1}{2n}.$$

**Solution:** We have  $n^2 < n^2 + 1 < n^2 + 2 < n^2 + 3 \cdots < n^2 + n$ . Hence we see that

$$\begin{aligned} \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} &> \frac{1}{n^2+n} + \frac{2}{n^2+n} + \cdots + \frac{n}{n^2+n} \\ &= \frac{1}{n^2+n} (1 + 2 + 3 + \cdots + n) = \frac{1}{2}. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} &< \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} \\ &= \frac{1}{n^2} (1 + 2 + 3 + \cdots + n) = \frac{1}{2} + \frac{1}{2n}. \end{aligned}$$

7. Find all integers  $a, b, c, d$  satisfying the following relations:

- (i)  $1 \leq a \leq b \leq c \leq d$ ;
- (ii)  $ab + cd = a + b + c + d + 3$ .

**Solution:** We may write (ii) in the form

$$ab - a - b + 1 + cd - c - d + 1 = 5.$$

Thus we obtain the equation  $(a-1)(b-1) + (c-1)(d-1) = 5$ . If  $a-1 \geq 2$ , then (i) shows that  $b-1 \geq 2$ ,  $c-1 \geq 2$  and  $d-1 \geq 2$  so that  $(a-1)(b-1) + (c-1)(d-1) \geq 8$ . It follows that  $a-1 = 0$  or  $1$ .

If  $a-1 = 0$ , then the contribution from  $(a-1)(b-1)$  to the sum is zero for any choice of  $b$ . But then  $(c-1)(d-1) = 5$  implies that  $c-1 = 1$  and  $d-1 = 5$  by (i). Again (i) shows that  $b-1 = 0$  or  $1$  since  $b \leq c$ . Taking  $b-1 = 0$ ,  $c-1 = 1$  and  $d-1 = 5$  we get the solution  $(a, b, c, d) = (1, 1, 2, 6)$ . Similarly,  $b-1 = 1$ ,  $c-1 = 1$  and  $d-1 = 5$  gives  $(a, b, c, d) = (1, 2, 2, 6)$ .

In the other case  $a-1 = 1$ , we see that  $b-1 = 2$  is not possible for then  $c-1 \geq 2$  and  $d-1 \geq 2$ . Thus  $b-1 = 1$  and this gives  $(c-1)(d-1) = 4$ . It follows that  $c-1 = 1$ ,  $d-1 = 4$  or  $c-1 = 2$ ,  $d-1 = 2$ . Considering each of these, we get two more solutions:  $(a, b, c, d) = (2, 2, 2, 5), (2, 2, 3, 3)$ .

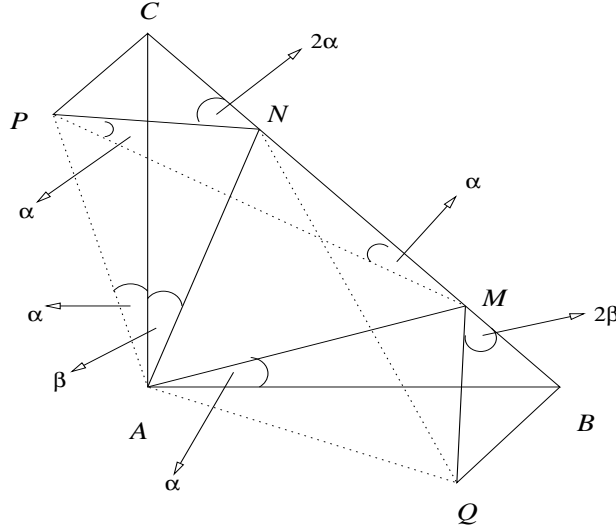
It is easy to verify all these four quadruples are indeed solutions to our problem.

## Solutions to CRMO-2003

1. Let  $ABC$  be a triangle in which  $AB = AC$  and  $\angle CAB = 90^\circ$ . Suppose  $M$  and  $N$  are points on the hypotenuse  $BC$  such that  $BM^2 + CN^2 = MN^2$ . Prove that  $\angle MAN = 45^\circ$ .

**Solution:**

Draw  $CP$  perpendicular to  $CB$  and  $BQ$  perpendicular to  $CB$  such that  $CP = BM$ ,  $BQ = CN$ . Join  $PA$ ,  $PM$ ,  $PN$ ,  $QA$ ,  $QM$ ,  $QN$ . (See Fig. 1.)



**Fig. 1.**

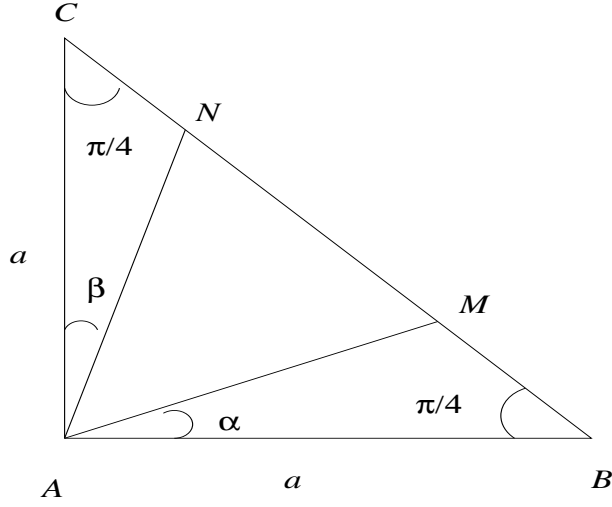
In triangles  $CPA$  and  $BMA$ , we have  $\angle PCA = 45^\circ = \angle MBA$ ;  $PC = MB$ ,  $CA = BA$ . So  $\triangle CPA \equiv \triangle BMA$ . Hence  $\angle PAC = \angle BAM = \alpha$ , say. Consequently,  $\angle MAP = \angle BAC = 90^\circ$ , whence  $PAMC$  is a cyclic quadrilateral. Therefore  $\angle PMC = \angle PAC = \alpha$ . Again  $PN^2 = PC^2 + CN^2 = BM^2 + CN^2 = MN^2$ . So  $PN = MN$ , giving  $\angle NPM = \angle NMP = \alpha$ , in  $\triangle PMN$ . Hence  $\angle PNC = 2\alpha$ . Likewise  $\angle QMB = 2\beta$ , where  $\beta = \angle CAN$ . Also  $\triangle NCP \equiv \triangle QBM$ , as  $CP = BM$ ,  $NC = BQ$  and  $\angle NCP = 90^\circ = \angle QBM$ . Therefore,  $\angle CPN = \angle BMQ = 2\beta$ , whence  $2\alpha + 2\beta = 90^\circ$ ;  $\alpha + \beta = 45^\circ$ ; finally  $\angle MAN = 90^\circ - (\alpha + \beta) = 45^\circ$ .

**Aliter:** Let  $AB = AC = a$ , so that  $BC = \sqrt{2}a$ ; and  $\angle MAB = \alpha$ ,  $\angle CAN = \beta$ . (See Fig. 2.)

By the Sine Law, we have from  $\triangle ABM$  that

$$\frac{BM}{\sin \alpha} = \frac{AB}{\sin(\alpha + 45^\circ)}.$$

So  $BM = \frac{a\sqrt{2}\sin\alpha}{\cos\alpha + \sin\alpha} = \frac{a\sqrt{2}u}{1+u}$ , where  $u = \tan\alpha$ .



**Fig. 2.**

Similarly  $CN = \frac{a\sqrt{2}v}{1+v}$ , where  $v = \tan\beta$ . But

$$\begin{aligned} BM^2 + CN^2 &= MN^2 = (BC - MB - NC)^2 \\ &= BC^2 + BM^2 + CN^2 \\ &\quad - 2BC \cdot MB - 2BC \cdot NC + MB \cdot NC. \end{aligned}$$

So

$$BC^2 - 2BC \cdot MB - 2BC \cdot NC + 2MB \cdot NC = 0.$$

This reduces to

$$2a^2 - 2\sqrt{2}a \frac{a\sqrt{2}u}{1+u} - 2\sqrt{2}a \frac{a\sqrt{2}v}{1+v} + \frac{4a^2uv}{(1+u)(1+v)} = 0.$$

Multiplying by  $(1+u)(1+v)/2a^2$ , we obtain

$$(1+u)(1+v) - 2u(1+v) - 2v(1+u) + 2uv = 0.$$

Simplification gives  $1 - u - v - uv = 0$ . So

$$\tan(\alpha + \beta) = \frac{u+v}{1-uv} = 1.$$

This gives  $\alpha + \beta = 45^\circ$ , whence  $\angle MAN = 45^\circ$ , as well.

2. If  $n$  is an integer greater than 7, prove that  $\binom{n}{7} - \left\lfloor \frac{n}{7} \right\rfloor$  is divisible by 7. [Here  $\binom{n}{7}$  denotes the number of ways of choosing 7 objects from among  $n$  objects; also, for any real number  $x$ ,  $\lfloor x \rfloor$  denotes the greatest integer not exceeding  $x$ .]

**Solution:** We have

$$\binom{n}{7} = \frac{n(n-1)(n-2)\dots(n-6)}{7!}.$$

In the numerator, there is a factor divisible by 7, and the other six factors leave the remainders 1,2,3,4,5,6 in some order when divided by 7.

Hence the numerator may be written as

$$7k \cdot (7k_1 + 1) \cdot (7k_2 + 2) \cdots (7k_6 + 6).$$

Also we conclude that  $\left\lfloor \frac{n}{7} \right\rfloor = k$ , as in the set  $\{n, n-1, \dots, n-6\}$ ,  $7k$  is the only number which is a multiple of 7. If the given number is called  $Q$ , then

$$\begin{aligned} Q &= 7k \cdot \frac{(7k_1 + 1)(7k_2 + 2) \cdots (7k_6 + 6)}{7!} - k \\ &= k \left[ \frac{(7k_1 + 1) \cdots (7k_6 + 6) - 6!}{6!} \right] \\ &= \frac{k[7t + 6! - 6!]}{6!} \\ &= \frac{7tk}{6!}. \end{aligned}$$

We know that  $Q$  is an integer, and so  $6!$  divides  $7tk$ . Since  $\gcd(7, 6!) = 1$ , even after cancellation there is a factor of 7 still left in the numerator. Hence 7 divides  $Q$ , as desired.

3. Let  $a, b, c$  be three positive real numbers such that  $a + b + c = 1$ . Prove that among the three numbers  $a - ab, b - bc, c - ca$  there is one which is at most  $1/4$  and there is one which is at least  $2/9$ .

**Solution:** By AM-GM inequality, we have

$$a(1-a) \leq \left( \frac{a+1-a}{2} \right)^2 = \frac{1}{4}.$$

Similarly we also have

$$b(1-b) \leq \frac{1}{4} \quad \text{and} \quad c(1-c) \leq \frac{1}{4}.$$

Multiplying these we obtain

$$abc(1-a)(1-b)(1-c) \leq \frac{1}{4^3}.$$

We may rewrite this in the form

$$a(1-b) \cdot b(1-c) \cdot c(1-a) \leq \frac{1}{4^3}.$$

Hence one factor at least (among  $a(1-b), b(1-c), c(1-a)$ ) has to be less than or equal to  $\frac{1}{4}$ ; otherwise **lhs** would exceed  $\frac{1}{4^3}$ .

Again consider the sum  $a(1-b)+b(1-c)+c(1-a)$ . This is equal to  $a+b+c-ab-bc-ca$ . We observe that

$$3(ab+bc+ca) \leq (a+b+c)^2,$$

which, in fact, is equivalent to  $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$ . This leads to the inequality

$$a+b+c-ab-bc-ca \geq (a+b+c) - \frac{1}{3}(a+b+c)^2 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence one summand at least (among  $a(1-b), b(1-c), c(1-a)$ ) has to be greater than or equal to  $\frac{2}{9}$ ; (otherwise **lhs** would be less than  $\frac{2}{3}$ .)

4. Find the number of ordered triples  $(x, y, z)$  of nonnegative integers satisfying the conditions:

- (i)  $x \leq y \leq z$ ;
- (ii)  $x + y + z \leq 100$ .

**Solution:** We count by brute force considering the cases  $x = 0, x = 1, \dots, x = 33$ . Observe that the least value  $x$  can take is zero, and its largest value is 33.

**x=0** If  $y = 0$ , then  $z \in \{0, 1, 2, \dots, 100\}$ ; if  $y=1$ , then  $z \in \{1, 2, \dots, 99\}$ ; if  $y = 2$ , then  $z \in \{2, 3, \dots, 98\}$ ; and so on. Finally if  $y = 50$ , then  $z \in \{50\}$ . Thus there are altogether  $101 + 99 + 97 + \dots + 1 = 51^2$  possibilities.

**x=1**. Observe that  $y \geq 1$ . If  $y = 1$ , then  $z \in \{1, 2, \dots, 98\}$ ; if  $y = 2$ , then  $z \in \{2, 3, \dots, 97\}$ ; if  $y = 3$ , then  $z \in \{3, 4, \dots, 96\}$ ; and so on. Finally if  $y = 49$ , then  $z \in \{49, 50\}$ . Thus there are altogether  $98 + 96 + 94 + \dots + 2 = 49 \cdot 50$  possibilities.

**General case.** Let  $x$  be even, say,  $x = 2k$ ,  $0 \leq k \leq 16$ . If  $y = 2k$ , then  $z \in \{2k, 2k+1, \dots, 100-4k\}$ ; if  $y = 2k+1$ , then  $z \in \{2k+1, 2k+2, \dots, 99-4k\}$ ; if  $y = 2k+2$ , then  $z \in \{2k+2, 2k+3, \dots, 99-4k\}$ ; and so on.

Finally, if  $y = 50 - k$ , then  $z \in \{50 - k\}$ . There are altogether

$$(101 - 6k) + (99 - 6k) + (97 - 6k) + \dots + 1 = (51 - 3k)^2$$

possibilities.

Let  $x$  be odd, say,  $x = 2k + 1$ ,  $0 \leq k \leq 16$ . If  $y = 2k + 1$ , then  $z \in \{2k + 1, 2k + 2, \dots, 98 - 4k\}$ ; if  $y = 2k + 2$ , then  $z \in \{2k + 2, 2k + 3, \dots, 97 - 4k\}$ ; if  $y = 2k + 3$ , then  $z \in \{2k + 3, 2k + 4, \dots, 96 - 4k\}$ ; and so on.

Finally, if  $y = 49 - k$ , then  $z \in \{49 - k, 50 - k\}$ . There are altogether

$$(98 - 6k) + (96 - 6k) + (94 - 6k) + \dots + 2 = (49 - 3k)(50 - 3k)$$

possibilities.

The last two cases would be as follows:

$x = 32$ : if  $y = 32$ , then  $z \in \{32, 33, 34, 35, 36\}$ ; if  $y = 33$ , then  $z \in \{33, 34, 35\}$ ; if  $y = 34$ , then  $z \in \{34\}$ ; altogether  $5 + 3 + 1 = 9 = 3^2$  possibilities.

$x = 33$ : if  $y = 33$ , then  $z \in \{33, 34\}$ ; only 2=1.2 possibilities.

Thus the total number of triples, say  $T$ , is given by,

$$T = \sum_{k=0}^{16} (51 - 3k)^2 + \sum_{k=0}^{16} (49 - 3k)(50 - 3k).$$

Writing this in the reverse order, we obtain

$$\begin{aligned} T &= \sum_{k=1}^{17} (3k)^2 + \sum_{k=0}^{17} (3k - 2)(3k - 1) \\ &= 18 \sum_{k=1}^{17} k^2 - 9 \sum_{k=1}^{17} k + 34 \\ &= 18 \left( \frac{17 \cdot 18 \cdot 35}{6} \right) - 9 \left( \frac{17 \cdot 18}{2} \right) + 34 \\ &= 30,787. \end{aligned}$$

Thus the answer is 30787.

### Aliter

It is known that the number of ways in which a given positive integer  $n \geq 3$  can be expressed as a sum of three positive integers  $x, y, z$  (that is,  $x + y + z = n$ ), subject to the condition  $x \leq y \leq z$  is  $\left\{ \frac{n^2}{12} \right\}$ , where  $\{a\}$  represents the integer closest to  $a$ . If

zero values are allowed for  $x, y, z$  then the corresponding count is  $\left\{ \frac{(n + 3)^2}{12} \right\}$ , where now  $n \geq 0$ .

Since in our problem  $n = x + y + z \in \{0, 1, 2, \dots, 100\}$ , the desired answer is

$$\sum_{n=0}^{100} \left\{ \frac{(n + 3)^2}{12} \right\}.$$

For  $n = 0, 1, 2, 3, \dots, 11$ , the corrections for  $\{ \}$  to get the nearest integers are

$$\frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}, \frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}.$$

So, for 12 consecutive integer values of  $n$ , the sum of the corrections is equal to

$$\left( \frac{3 - 4 - 1 - 0 - 1 - 4 - 3}{12} \right) \times 2 = \frac{-7}{6}.$$

Since  $\frac{101}{12} = 8 + \frac{5}{12}$ , there are 8 sets of 12 consecutive integers in  $\{3, 4, 5, \dots, 103\}$  with 99, 100, 101, 102, 103 still remaining. Hence the total correction is

$$\left( \frac{-7}{6} \right) \times 8 + \frac{3 - 4 - 1 - 0 - 1}{12} = \frac{-28}{3} - \frac{1}{4} = \frac{-115}{12}.$$

So the desired number  $T$  of triples  $(x, y, z)$  is equal to

$$\begin{aligned} T &= \sum_{n=0}^{100} \frac{(n+3)^2}{12} - \frac{115}{12} \\ &= \frac{(1^2 + 2^2 + 3^2 + \dots + 103^2) - (1^2 + 2^2)}{12} - \frac{115}{12} \\ &= \frac{103 \cdot 104 \cdot 207}{6 \cdot 12} - \frac{5}{12} - \frac{115}{12} \\ &= 30787. \end{aligned}$$

5. Suppose  $P$  is an interior point of a triangle  $ABC$  such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \quad \frac{d(B, CA)}{d(P, CA)}, \quad \frac{d(C, AB)}{d(P, AB)}$$

are all equal. Find the common value of these ratios. [Here  $d(X, YZ)$  denotes the perpendicular distance from a point  $X$  to the line  $YZ$ .]

**Solution:** Let  $AP, BP, CP$  when extended, meet the sides  $BC, CA, AB$  in  $D, E, F$  respectively. Draw  $AK, PL$  perpendicular to  $BC$  with  $K, L$  on  $BC$ . (See Fig. 3.)

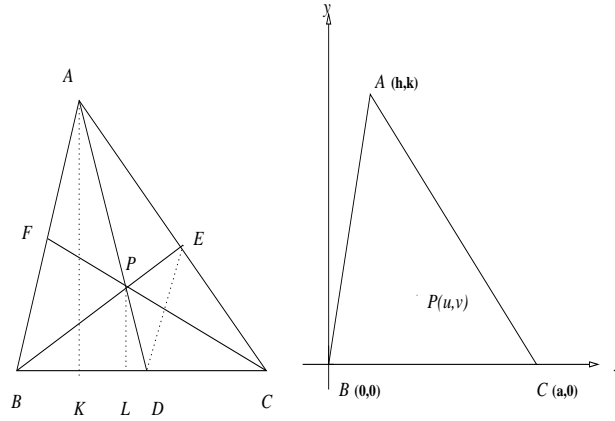


Fig. 3.

Fig. 4.

Now

$$\frac{d(A, BC)}{d(P, BC)} = \frac{AK}{PL} = \frac{AD}{PD}.$$

Similarly,

$$\frac{d(B, CA)}{d(P, CA)} = \frac{BE}{PE} \quad \text{and} \quad \frac{d(C, AB)}{d(P, AB)} = \frac{CF}{PF}.$$

So, we obtain

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF}, \quad \text{and hence} \quad \frac{AP}{PD} = \frac{BP}{PE} = \frac{CP}{PF}.$$

From  $\frac{AP}{PD} = \frac{BP}{PE}$  and  $\angle APB = \angle DPE$ , it follows that triangles  $APB$  and  $DPE$  are similar. So  $\angle ABP = \angle DEP$  and hence  $AB$  is parallel to  $DE$ .

Similarly,  $BC$  is parallel to  $EF$  and  $CA$  is parallel to  $DF$ . Using these we obtain

$$\frac{BD}{DC} = \frac{AE}{EC} = \frac{AF}{FB} = \frac{DC}{BD},$$

whence  $BD^2 = CD^2$  or which is same as  $BD = CD$ . Thus  $D$  is the midpoint of  $BC$ . Similarly  $E, F$  are the midpoints of  $CA$  and  $AB$  respectively.

We infer that  $AD, BE, CF$  are indeed the medians of the triangle  $ABC$  and hence  $P$  is the centroid of the triangle. So

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF} = 3,$$

and consequently each of the given ratios is also equal to 3.

**Aliter**

Let  $ABC$ , the given triangle be placed in the  $xy$ -plane so that  $B = (0, 0), C = (a, 0)$  (on the  $x$ - axis). (See Fig. 4.)

Let  $A = (h, k)$  and  $P = (u, v)$ . Clearly  $d(A, BC) = k$  and  $d(P, BC) = v$ , so that

$$\frac{d(A, BC)}{d(P, BC)} = \frac{k}{v}.$$

The equation to  $CA$  is  $kx - (h - a)y - ka = 0$ . So

$$\begin{aligned} \frac{d(B, CA)}{d(P, CA)} &= \frac{-ka}{\sqrt{k^2 + (h - a)^2}} \bigg/ \frac{(ku - (h - a)v - ka)}{\sqrt{k^2 + (h - a)^2}} \\ &= \frac{-ka}{ku - (h - a)v - ka}. \end{aligned}$$

Again the equation to  $AB$  is  $kx - hy = 0$ . Therefore

$$\begin{aligned} \frac{d(C, AB)}{d(P, AB)} &= \frac{ka}{\sqrt{h^2 + k^2}} \bigg/ \frac{(ku - hv)}{\sqrt{h^2 + k^2}} \\ &= \frac{ka}{ku - hv}. \end{aligned}$$

From the equality of these ratios, we get

$$\frac{k}{v} = \frac{-ka}{ku - (h - a)v - ka} = \frac{ka}{ku - hv}.$$

The equality of the first and third ratios gives  $ku - (h + a)v = 0$ . Similarly the equality of second and third ratios gives  $2ku - (2h - a)v = ka$ . Solving for  $u$  and  $v$ , we get

$$u = \frac{h + a}{3}, \quad v = \frac{k}{3}.$$

Thus  $P$  is the centroid of the triangle and each of the ratios is equal to  $\frac{k}{v} = 3$ .

6. Find all real numbers  $a$  for which the equation

$$x^2 + (a - 2)x + 1 = 3|x|$$

has exactly three distinct real solutions in  $x$ .

**Solution:** If  $x \geq 0$ , then the given equation assumes the form,

$$x^2 + (a - 5)x + 1 = 0. \quad \dots(1)$$

If  $x < 0$ , then it takes the form

$$x^2 + (a + 1)x + 1 = 0. \quad \dots(2)$$

For these two equations to have exactly three distinct real solutions we should have

(I) either  $(a - 5)^2 > 4$  and  $(a + 1)^2 = 4$ ;

(II) or  $(a - 5)^2 = 4$  and  $(a + 1)^2 > 4$ .

**Case (I)** From  $(a + 1)^2 = 4$ , we have  $a = 1$  or  $-3$ . But only  $a = 1$  satisfies  $(a - 5)^2 > 4$ . Thus  $a = 1$ . Also when  $a = 1$ , equation (1) has solutions  $x = 2 + \sqrt{3}$ ; and (2) has solutions  $x = -1, -1$ . As  $2 + \sqrt{3} > 0$  and  $-1 < 0$ , we see that  $a = 1$  is indeed a solution.

**Case (II)** From  $(a - 5)^2 = 4$ , we have  $a = 3$  or  $7$ . Both these values of  $a$  satisfy the inequality  $(a + 1)^2 > 4$ . When  $a = 3$ , equation (1) has solutions  $x = 1, 1$  and (2) has the solutions  $x = -2 \pm \sqrt{3}$ . As  $1 > 0$  and  $-2 \pm \sqrt{3} < 0$ , we see that  $a = 3$  is in fact a solution.

When  $a = 7$ , equation (1) has solutions  $x = -1, -1$ , which are negative contradicting  $x \geq 0$ .

Thus  $a = 1, a = 3$  are the two desired values.

7. Consider the set  $X = \{1, 2, 3, \dots, 9, 10\}$ . Find two disjoint nonempty subsets  $A$  and  $B$  of  $X$  such that

- (a)  $A \cup B = X$ ;
- (b)  $\text{prod}(A)$  is divisible by  $\text{prod}(B)$ , where for any finite set of numbers  $C$ ,  $\text{prod}(C)$  denotes the product of all numbers in  $C$  ;
- (c) the quotient  $\text{prod}(A)/\text{prod}(B)$  is as small as possible.

**Solution:** The prime factors of the numbers in set  $\{1, 2, 3, \dots, 9, 10\}$  are 2, 3, 5, 7. Also only  $7 \in X$  has the prime factor 7. Hence it cannot appear in  $B$ . For otherwise, 7 in the denominator would not get canceled. Thus  $7 \in A$ .

Hence

$$\text{prod}(A)/\text{prod}(B) \geq 7.$$

The numbers having prime factor 3 are 3, 6, 9. So 3 and 6 should belong to one of  $A$  and  $B$ , and 9 belongs to the other. We may take  $3, 6 \in A, 9 \in B$ .

Also 5 divides 5 and 10. We take  $5 \in A, 10 \in B$ . Finally we take  $1, 2, 4 \in A, 8 \in B$ . Thus

$$A = \{1, 2, 3, 4, 5, 6, 7\}, \quad B = \{8, 9, 10\},$$

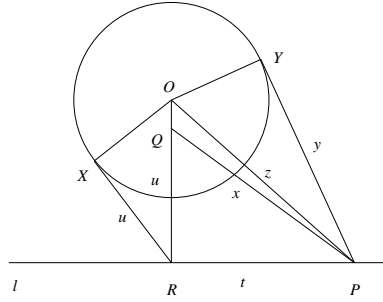
so that

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{8 \cdot 9 \cdot 10} = 7.$$

Thus 7 is the minimum value of  $\frac{\text{prod}(A)}{\text{prod}(B)}$ . There are other possibilities for  $A$  and  $B$ : e.g., 1 may belong to either  $A$  or  $B$ . We may take  $A = \{3, 5, 6, 7, 8\}$ ,  $B = \{1, 2, 4, 9, 10\}$ .

1. Consider in the plane a circle  $\Gamma$  with center  $O$  and a line  $l$  not intersecting circle  $\Gamma$ . Prove that there is a unique point  $Q$  on the perpendicular drawn from  $O$  to the line  $l$ , such that for any point  $P$  on the line  $l$ ,  $PQ$  represents the length of the tangent from  $P$  to the circle  $\Gamma$ .

**Solution:**



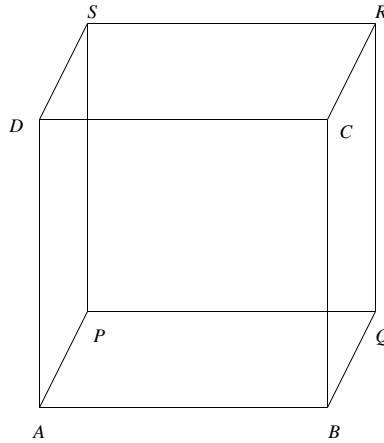
Let  $R$  be the foot of the perpendicular from  $O$  to the line  $l$ , and  $u$  be the length of the tangent  $RX$  from  $R$  to circle  $\Gamma$ . On  $OR$  take a point  $Q$  such that  $QR = u$ . We show that  $Q$  is the desired point. To this end, take any point  $P$  on line  $l$  and let  $y$  be the length of the tangent  $PY$  from  $P$  to  $\Gamma$ .

Further let  $r$  be the radius of the circle  $\Gamma$  and let  $y$  be the length of the tangent  $PY$  from  $P$  to  $\Gamma$ . Join  $OP, QP$ . Let  $QP = x, OP = z, RP = t$ . From right angled triangles  $POY, OXP, ORP, PQR$  we have respectively  $z^2 = r^2 + y^2, OR^2 = r^2 + u^2, z^2 = OR^2 + t^2 = r^2 + u^2 + t^2, x^2 = u^2 + t^2$ . So we obtain  $y^2 = z^2 - r^2 = r^2 + u^2 + t^2 - r^2 = u^2 + t^2 = x^2$ . Hence  $y = x$ . This gives  $PY = PX$  which is what we needed to show.

2. Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If  $T$  denotes the sum of the numbers on all the faces, find all the possible values of  $T$ .

**Solution:**

Let  $ABCDPQRS$  be a cube, and the numbers  $a, b, c, d, e, f$  be written on the faces  $ABCD, BQRC, PQRS, APSD, ABQP, CRSD$  respectively. Then the products written at the corners  $A, B, C, D, P, Q, R, S$  are respectively  $ade, abe, abf, adf, cde, bce, bcf, cdf$ . The sum of these 8 numbers is:



$$= (e+f)(ab+bc+cd+ad) = (e+f)(a+c)(b+d).$$

This is given to be equal to  $2004 = 2^2 \cdot 3 \cdot 167$ . Observe that none of the factors  $a+c$ ,  $b+d$ ,  $e+f$  is equal to 1. Thus  $(a+c)(b+d)(e+f)$  is equal to  $4 \cdot 3 \cdot 167$ ,  $2 \cdot 6 \cdot 167$ ,  $2 \cdot 3 \cdot 334$  or  $2 \cdot 2 \cdot 501$ . Hence the possible values of  $T = a+b+c+d+e+f$  are  $4+3+167=174$ ,  $2+6+167=175$ ,  $2+3+334=339$ , or  $2+2+501=505$ .

Thus there are 4 possible values of  $T$  and they are 174, 175, 339, 505.

3. Let  $\alpha$  and  $\beta$  be the roots of the quadratic equation  $x^2 + mx - 1 = 0$ , where  $m$  is an odd integer. Let  $\lambda_n = \alpha^n + \beta^n$ , for  $n \geq 0$ . Prove that for  $n \geq 0$ ,

(a)  $\lambda_n$  is an integer; and

(b)  $\gcd(\lambda_n, \lambda_{n+1}) = 1$ .

**Solution:** Since  $\alpha$  and  $\beta$  are the roots of the equation  $x^2 + mx - 1 = 0$ , we have  $\alpha^2 + m\alpha - 1 = 0$ ,  $\beta^2 + m\beta - 1 = 0$ . Multiplying by  $\alpha^{n-2}$ ,  $\beta^{n-2}$  respectively we have  $\alpha^n + m\alpha^{n-1} - \alpha^{n-2} = 0$  and  $\beta^n + m\beta^{n-1} - \beta^{n-2} = 0$ .

Adding we obtain  $\alpha^n + \beta^n = -m(\alpha^{n-1} + \beta^{n-1}) + (\alpha^{n-2} + \beta^{n-2})$ . This gives a recurrence relation for  $n \geq 2$ :

$$\lambda_n = -m\lambda_{n-1} + \lambda_{n-2}, n \geq 2 \quad (\star)$$

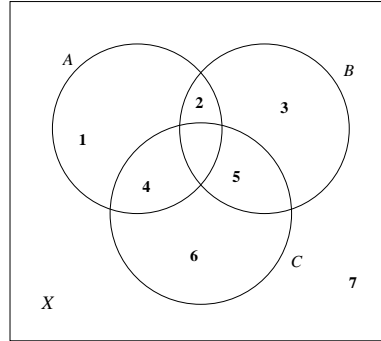
(a) Now  $\lambda_0 = 1 + 1 = 2$  and  $\lambda_1 = \alpha + \beta = -m$ . Thus  $\lambda_0$  and  $\lambda_1$  are integers. By induction, it follows from  $(\star)$  that  $\lambda_n$  is an integer for each  $n \geq 0$ .

(b) We again use  $(\star)$  to prove by induction that  $\gcd(\lambda_n, \lambda_{n+1}) = 1$ . This is clearly true for  $n = 0$ , as  $\gcd(2, -m) = 1$ , by the given condition that  $m$  is odd.

Let  $\gcd(\lambda_{n-2}, \lambda_{n-1}) = 1$ ,  $n \geq 2$ . If it were to happen that  $\gcd(\lambda_{n-1}, \lambda_n) > 1$ , take a prime  $p$  that divides both  $\lambda_{n-1}$  and  $\lambda_n$ . Then from  $(\star)$ , we get that  $p$  divides  $\lambda_{n-2}$  also. Thus  $p$  is a factor of  $\gcd(\lambda_{n-2}, \lambda_{n-1})$ , a contradiction. So  $\gcd(\lambda_{n-1}, \lambda_n) = 1$ . Hence we have  $\gcd(\lambda_n, \lambda_{n+1}) = 1$ , for all  $n \geq 0$ .

4. Prove that the number of triples  $(A, B, C)$  where  $A, B, C$  are subsets of  $\{1, 2, \dots, n\}$  such that  $A \cap B \cap C = \emptyset$ ,  $A \cap B \neq \emptyset$ ,  $B \cap C \neq \emptyset$  is  $7^n - 2 \cdot 6^n + 5^n$ .

**Solution:**



Let  $X = \{1, 2, 3, \dots, n\}$ . We use Venn diagram for sets  $A, B, C$  to solve the problem. The regions other than  $A \cap B \cap C$  (which is to be empty) are numbered 1, 2, 3, 4, 5, 6, 7 as shown in the figure; e.g., 1 corresponds to  $A \setminus (B \cup C) = A \cap B^c \cap C^c$ , 2 corresponds to  $A \cap B \setminus C = A \cap B \cap C^c$ , 7 corresponds to  $X \setminus (A \cup B \cup C) = A^c \cap B^c \cap C^c$ , since  $A \cap B \cap C = \emptyset$ .

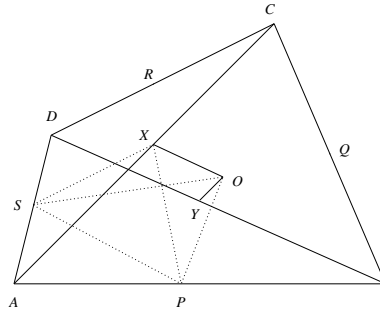
Firstly the number of ways of assigning elements of  $X$  to the numbers regions without any condition is  $7^n$ . Among these there are cases in which 2 or 5 or both are empty. The number of distributions in which 2 is empty is  $6^n$ . Likewise the number of distributions in which 5 is empty is also  $6^n$ . But then we have subtracted twice the number of distributions in which both the regions 2 and 5 are empty. So to compensate we have to add the number of distributions in which both 2 and 5 are empty. This is  $5^n$ . Hence the desired number of triples  $(A, B, C)$  in  $7^n - 6^n - 6^n + 5^n = 7^n - 2 \cdot 6^n + 5^n$ .

5. Let  $ABCD$  be a quadrilateral;  $x$  and  $Y$  be the midpoints of  $AC$  and  $BD$  respectively and the lines through  $X$  and  $Y$  respectively parallel to  $BD, AC$  meet in  $O$ . Let  $P, Q, R, S$  be the midpoints of  $AB, BC, CD, DA$  respectively. Prove that

- (a) quadrilaterals  $APOS$  and  $APXS$  have the same area;
- (b) the areas of the quadrilateral  $APOS, BQOP, CROQ, DSOR$  are all equal.

**Solution:**

We use the facts: (i) the line joining the midpoints of the sides of a triangle is parallel to the third side; (ii) any median of a triangle bisects its area; (iii) two triangles having equal bases and bounded by same parallel lines have equal area.



- (a) Now  $BD$  is parallel to  $PS$  as well as  $OX$ . So  $OX$  is parallel to  $PS$ . Hence  $[PXS] = [POS]$ . Adding  $[PAS]$  to both sides we get  $[APXS] = [APOS]$ . This proves part (a).
- (b) Now

$$\begin{aligned}
 [APXS] &= [APX] + [ASX] \\
 &= \frac{1}{2}[ABX] + \frac{1}{2}[ADX] = \frac{1}{4}[ABC] + \frac{1}{4}[ADC] \\
 &= \frac{1}{4}[ABCD].
 \end{aligned}$$

Hence by (a),  $[APOS] = \frac{1}{4}[ABCD]$ . Similarly by symmetry each of the areas  $[AQOP], [CROQ]$  and  $[DSOR]$  is equal to  $\frac{1}{4}[ABCD]$ . Thus the four given areas are equal. This proves part (b). [Note:  $[\ ]$  denotes area].

6. Let  $\langle p_1, p_2, p_3, \dots, p_n, \dots \rangle$  be a sequence of primes defined by  $p_1 = 2$  and for  $n \geq 1, p_{n+1}$  is the largest prime factor of  $p_1 p_2 \dots p_n + 1$ . (Thus  $p_2 = 3, p_3 = 7$ ). Prove that  $p_n \neq 5$  for any  $n$ .

**Solution:** By data  $p_1 = 2, p_2 = 3, p_3 = 7$ . It follows by induction that  $p_n, n \geq 2$  is odd. [For if  $p_2, p_3, \dots, p_{n-1}$  are odd, then  $p_1 p_2 \dots p_{n-1} + 1$  is also odd and not 3. This also follows by induction. For if  $p_3 = 7$  and if  $p_3, p_4, \dots, p_{n-1}$  are neither 2 nor 3, then  $p_1 p_2 p_3 \dots p_{n-1} + 1$  are neither 2 nor 3. So  $p_n$  is neither 2 nor 3.

7. Let  $x$  and  $y$  be positive real numbers such that  $y^3 + y \leq x - x^3$ . Prove that

(a)  $y < x < 1$ ; and

(b)  $x^2 + y^2 < 1$ .

**Solution:**

(a) Since  $x$  and  $y$  are positive, we have  $y \leq x - x^3 - y^3 < x$ . Also  $x - x^3 \geq y + y^3 > 0$ . So  $x(1 - x^2) > 0$ . Hence  $x < 1$ . Thus  $y < x < 1$ , proving part (a).

(b) Again  $x^3 + y^3 \leq x - y$ . So

$$x^2 - xy + y^2 \leq \frac{x - y}{x + y}.$$

That is

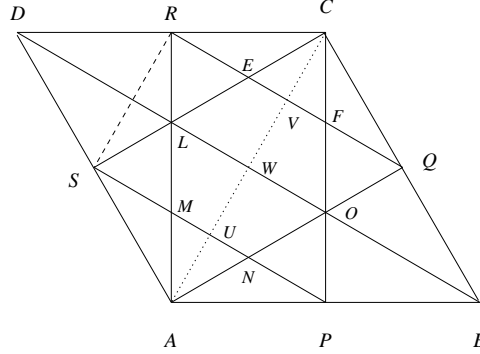
$$x^2 + y^2 \leq \frac{x - y}{x + y} + xy = \frac{x - y + xy(x + y)}{x + y}.$$

Here  $xy(x + y) < 1 \cdot y \cdot (1 + 1) = 2y$ . So  $x^2 + y^2 < \frac{x - y + 2y}{x + y} = \frac{x + y}{x + y} = 1$ . This proves (b).

## Problems and Solutions of CRMO-2005

1. Let  $ABCD$  be a convex quadrilateral;  $P, Q, R, S$  be the midpoints of  $AB, BC, CD, DA$  respectively such that triangles  $AQR$  and  $CSP$  are equilateral. Prove that  $ABCD$  is a rhombus. Determine its angles.

**Solution:** We have  $QR = BD/2 = PS$ . Since  $AQR$  and  $CSP$  are both equilateral and  $QR = PS$ , they must be congruent triangles. This implies that  $AQ = QR = RA = CS = SP = PC$ . Also  $\angle CEF = 60^\circ = \angle RQA$ . (See Fig. 1.)



**Fig. 1.**

Hence  $CS$  is parallel to  $QA$ . Now  $CS = QA$  implies that  $CSQA$  is a parallelogram. In particular  $SA$  is parallel to  $CQ$  and  $SA = CQ$ . This shows that  $AD$  is parallel to  $BC$  and  $AD = BC$ . Hence  $ABCD$  is a parallelogram.

Let the diagonal  $AC$  and  $BD$  bisect each other at  $W$ . Then  $DW = DB/2 = QR = CS = AR$ . Thus in triangle  $ADC$ , the medians  $AR, DW, CS$  are all equal. Thus  $ADC$  is equilateral. This implies  $ABCD$  is a rhombus. Moreover the angles are  $60^\circ$  and  $120^\circ$ .

2. If  $x, y$  are integers, and 17 divides both the expressions  $x^2 - 2xy + y^2 - 5x + 7y$  and  $x^2 - 3xy + 2y^2 + x - y$ , then prove that 17 divides  $xy - 12x + 15y$ .

**Solution:** Observe that  $x^2 - 3xy + 2y^2 + x - y = (x - y)(x - 2y + 1)$ . Thus 17 divides either  $x - y$  or  $x - 2y + 1$ . Suppose that 17 divides  $x - y$ . In this case  $x \equiv y \pmod{17}$  and hence

$$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 2y^2 + y^2 - 5y + 7y \equiv 2y \pmod{17}.$$

Thus the given condition that 17 divides  $x^2 - 2xy + y^2 - 5x + 7y$  implies that 17 also divides  $2y$  and hence  $y$  itself. But then  $x \equiv y \pmod{17}$  implies that 17 divides  $x$  also. Hence in this case 17 divides  $xy - 12x + 15y$ .

Suppose on the other hand that 17 divides  $x - 2y + 1$ . Thus  $x \equiv 2y - 1 \pmod{17}$  and hence

$$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 5y + 6 \pmod{17}.$$

Thus 17 divides  $y^2 - 5y + 6$ . But  $x \equiv 2y - 1 \pmod{17}$  also implies that

$$xy - 12x + 15y \equiv 2(y^2 - 5y + 6) \pmod{17}.$$

Since 17 divides  $y^2 - 5y + 6$ , it follows that 17 divides  $xy - 12x + 15y$ .

3. If  $a, b, c$  are three real numbers such that  $|a - b| \geq |c|$ ,  $|b - c| \geq |a|$ ,  $|c - a| \geq |b|$ , then prove that one of  $a, b, c$  is the sum of the other two.

**Solution:** Using  $|a - b| \geq |c|$ , we obtain  $(a - b)^2 \geq c^2$  which is equivalent to  $(a - b - c)(a - b + c) \geq 0$ . Similarly,  $(b - c - a)(b - c + a) \geq 0$  and  $(c - a - b)(c - a + b) \geq 0$ . Multiplying these inequalities, we get

$$-(a + b - c)^2(b + c - a)^2(c + a - b)^2 \geq 0.$$

This forces that **lhs** is equal to zero. Hence it follows that either  $a + b = c$  or  $b + c = a$  or  $c = a = b$ .

4. Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, 31545, 34155 are two such numbers.)

**Solution:** Any such number should be both divisible by 5 and 3. The last digit of a number divisible by 5 must be either 5 or 0. Hence any such number falls into one of the following seven categories:

(i)  $abc15$ ; (ii)  $ab150$ ; (iii)  $ab155$ ; (iv)  $a15b0$ ; (v)  $a15b5$ ; (vi)  $15ab0$ ; (vii)  $15ab5$ .

Here  $a, b, c$  are digits. Let us count how many numbers of each category are there.

(i) In this case  $a \neq 0$ , and the 3-digit number  $abc$  is divisible by 3, and hence one of the numbers in the set  $\{102, 105, \dots, 999\}$ . This gives 300 numbers.

(ii) Again a number of the form  $ab150$  is divisible by 15 if and only if the 2-digit number  $ab$  is divisible by 3. Hence it must be from the set  $\{12, 15, \dots, 99\}$ . There are 30 such numbers.

(iii) As in (ii), here are again 30 numbers.

(iv) Similar to (ii); 30 numbers.

(v) Similar to (ii), 30 numbers.

(vi) We can begin the analysis of the number of the form  $15ab0$  as in (ii). Here again  $ab$  as a 2-digit number must be divisible by 3, but  $a = 0$  is also permissible. Hence it must be from the set  $\{00, 03, 06, \dots, 99\}$ . There are 34 such numbers.

(vii) Here again there are 33 numbers;  $ab$  must be from the set  $\{01, 04, 07, \dots, 97\}$ .

Adding all these we get  $300 + 30 + 30 + 30 + 30 + 34 + 33 = 487$  numbers.

However this is not the correct figure as there is over counting. Let us see how much over counting is done by looking at the intersection of each pair of categories. A number in (i) obviously cannot lie in (ii), (iv) or (vi) as is evident from the last digit. There cannot be a common number in (i) and (iii) as any two such numbers differ in the 4-th digit. If a number belongs to both (i) and (v), then such a number of the form  $a1515$ . This is divisible by 3 only for  $a = 3, 6, 9$ . Thus there are 3 common numbers in (i) and (ii). A number which is both in (i) and (vii) is of the form  $15c15$  and divisibility by 3 gives  $c = 0, 3, 6, 9$ ; thus we have 4 numbers common in (i) and (vii). That exhaust all possibilities with (i).

Now (ii) can have common numbers with only categories (iv) and (vi). There are no numbers common between (ii) and (vi) as evident from 3-rd digit. There is only one number common to (ii) and (vi), namely  $15150$  and this is divisible by 3. There is nothing common to (iii) and (v) as can be seen from the 3-rd digit. The only number common to (iii) and (vii) is  $15155$  and this is not divisible by 3. It can easily be inferred that no number is common to (iv) and (vi) by looking at the 2-nd digit. Similarly no number is common to (v) and (vii). Thus there are  $3+4+1=8$  numbers which are counted twice.

We conclude that the number of 5-digit numbers which contain the block 15 and divisible by 15 is  $487 - 8 = 479$ .

5. In triangle  $ABC$ , let  $D$  be the midpoint of  $BC$ . If  $\angle ADB = 45^\circ$  and  $\angle ACD = 30^\circ$ , determine  $\angle BAD$ .

**Solution:** Draw  $BL$  perpendicular to  $AC$  and join  $L$  to  $D$ . (See Fig. 2.)

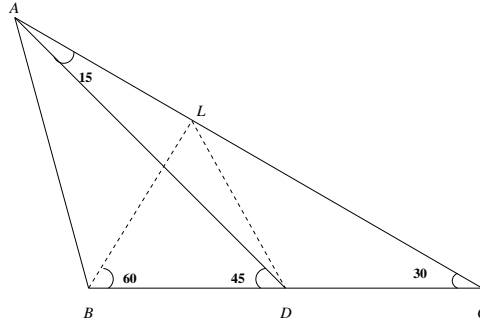


Fig. 2.

Since  $\angle BCL = 30^\circ$ , we get  $\angle CBL = 60^\circ$ . Since  $BLC$  is a right-triangle with  $\angle BCL = 30^\circ$ , we have  $BL = BC/2 = BD$ . Thus in triangle  $BLD$ , we observe that  $BL = BD$  and  $\angle DBL = 60^\circ$ . This implies that  $BLD$  is an equilateral triangle and hence  $LB = LD$ . Using  $\angle LDB = 60^\circ$  and  $\angle ADB = 45^\circ$ , we get  $\angle ADL = 15^\circ$ . But  $\angle DAL = 15^\circ$ . Thus  $LD = LA$ . We hence have  $LD = LA = LB$ . This implies that  $L$  is the circumcentre of the triangle  $BDA$ . Thus

$$\angle BAD = \frac{1}{2}\angle BLD = \frac{1}{2} \times 60^\circ = 30^\circ.$$

6. Determine all triples  $(a, b, c)$  of positive integers such that  $a \leq b \leq c$  and

$$a + b + c + ab + bc + ca = abc + 1.$$

**Solution:** Putting  $a - 1 = p$ ,  $b - 1 = q$  and  $c - 1 = r$ , the equation may be written in the form

$$pqr = 2(p + q + r) + 4,$$

where  $p, q, r$  are integers such that  $0 \leq p \leq q \leq r$ . Observe that  $p = 0$  is not possible, for then  $0 = 2(p + q) + 4$  which is impossible in nonnegative integers. Thus we may write this in the form

$$2\left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right) + \frac{4}{pqr} = 1.$$

If  $p \geq 3$ , then  $q \geq 3$  and  $r \geq 3$ . Then left side is bounded by  $6/9 + 4/27$  which is less than 1. We conclude that  $p = 1$  or 2.

**Case 1.** Suppose  $p = 1$ . Then we have  $qr = 2(q + r) + 6$  or  $(q - 2)(r - 2) = 10$ . This gives  $q - 2 = 1$ ,  $r - 2 = 10$  or  $q - 2 = 2$  and  $r - 2 = 5$  (recall  $q \leq r$ ). This implies  $(p, q, r) = (1, 3, 12)$ ,  $(1, 4, 7)$ .

**Case 2.** If  $p = 2$ , the equation reduces to  $2qr = 2(2 + q + r) + 4$  or  $qr = q + r + 4$ . This reduces to  $(q - 1)(r - 1) = 5$ . Hence  $q - 1 = 1$  and  $r - 1 = 5$  is the only solution. This gives  $(p, q, r) = (2, 2, 6)$ .

Reverting back to  $a, b, c$ , we get three triples:  $(a, b, c) = (2, 4, 13), (2, 5, 8), (3, 3, 7)$ .

7. Let  $a, b, c$  be three positive real numbers such that  $a + b + c = 1$ . Let

$$\lambda = \min \{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}.$$

Prove that the roots of the equation  $x^2 + x + 4\lambda = 0$  are real.

**Solution:** Suppose the equation  $x^2 + x + 4\lambda = 0$  has no real roots. Then  $1 - 16\lambda < 0$ . This implies that

$$1 - 16(a^3 + a^2bc) < 0, \quad 1 - 16(b^3 + ab^2c) < 0, \quad 1 - 16(c^3 + abc^2) < 0.$$

Observe that

$$\begin{aligned} 1 - 16(a^3 + a^2bc) < 0 &\implies 1 - 16a^2(a + bc) < 0 \\ &\implies 1 - 16a^2(1 - b - c + bc) < 0 \\ &\implies 1 - 16a^2(1 - b)(1 - c) < 0 \\ &\implies \frac{1}{16} < a^2(1 - b)(1 - c). \end{aligned}$$

Similarly we may obtain

$$\frac{1}{16} < b^2(1 - c)(1 - a), \quad \frac{1}{16} < c^2(1 - a)(1 - b).$$

Multiplying these three inequalities, we get

$$a^2b^2c^2(1 - a)^2(1 - b)^2(1 - c)^2 > \frac{1}{16^3}.$$

However,  $0 < a < 1$  implies that  $a(1 - a) \leq 1/4$ . Hence

$$a^2b^2c^2(1 - a)^2(1 - b)^2(1 - c)^2 = (a(1 - a))^2(b(1 - b))^2(c(1 - c))^2 \leq \frac{1}{16^3},$$

a contradiction. We conclude that the given equation has real roots.

---

- Let  $ABC$  be an acute-angled triangle and let  $D, E, F$  be the feet of perpendiculars from  $A, B, C$  respectively to  $BC, CA, AB$ . Let the perpendiculars from  $F$  to  $CB, CA, AD, BE$  meet them in  $P, Q, M, N$  respectively. Prove that  $P, Q, M, N$  are *collinear*.

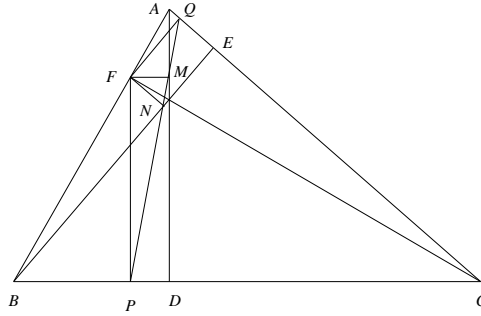
**Solution:** Observe that  $C, Q, F, P$  are concyclic. Hence

$$\angle CQP = \angle CFP = 90^\circ - \angle FCP = \angle B.$$

Similarly the concyclicity of  $F, M, Q, A$  gives

$$\angle AQN = 90^\circ + \angle FQM = 90^\circ + \angle FAM = 90^\circ + 90^\circ - \angle B = 180^\circ - \angle B.$$

Thus we obtain  $\angle CQP + \angle AQN = 180^\circ$ . It follows that  $Q, N, P$  lie on the same line.



We can similarly prove that  $\angle CPQ + \angle BPM = 180^\circ$ . This implies that  $P, M, Q$  are collinear. Thus  $M, N$  both lie on the line joining  $P$  and  $Q$ .

- Find the *least* possible value of  $a + b$ , where  $a, b$  are positive integers such that 11 divides  $a + 13b$  and 13 divides  $a + 11b$ .

**Solution:** Since 13 divides  $a + 11b$ , we see that 13 divides  $a - 2b$  and hence it also divides  $6a - 12b$ . This in turn implies that  $13|(6a + b)$ . Similarly  $11|(a + 13b) \implies 11|(a + 2b) \implies 11|(6a + 12b) \implies 11|(6a + b)$ . Since  $\gcd(11, 13) = 1$ , we conclude that  $143|(6a + b)$ . Thus we may write  $6a + b = 143k$  for some natural number  $k$ . Hence

$$6a + 6b = 143k + 5b = 144k + 6b - (k + b).$$

This shows that 6 divides  $k + b$  and hence  $k + b \geq 6$ . We therefore obtain

$$6(a + b) = 143k + 5b = 138k + 5(k + b) \geq 138 + 5 \times 6 = 168.$$

It follows that  $a + b \geq 28$ . Taking  $a = 23$  and  $b = 5$ , we see that the conditions of the problem are satisfied. Thus the minimum value of  $a + b$  is 28.

- If  $a, b, c$  are three positive real numbers, prove that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

**Solution:** We use the trivial inequalities  $a^2 + 1 \geq 2a$ ,  $b^2 + 1 \geq 2b$  and  $c^2 + 1 \geq 2c$ . Hence we obtain

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq \frac{2a}{b + c} + \frac{2b}{c + a} + \frac{2c}{a + b}.$$

$$\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \geq 3.$$

Adding 6 both sides, this is equivalent to

$$(2a + 2b + 2c) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \geq 9.$$

Taking  $x = b + c$ ,  $y = c + a$ ,  $z = a + b$ , this is equivalent to

$$(x + y + z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 9.$$

This is a consequence of AM-GM inequality.

**Alternately:** The substitutions  $b + c = x$ ,  $c + a = y$ ,  $a + b = z$  leads to

$$\sum \frac{2a}{b+c} = \sum \frac{y+z-x}{x} = \sum \left( \frac{y}{x} + \frac{z}{x} \right) - 3 \geq 6 - 3 = 3.$$

4. A  $6 \times 6$  square is dissected in to 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides. Prove that there are always **two** congruent rectangles.

**Solution:** Consider the dissection of the given  $6 \times 6$  square in to non-congruent rectangles with least possible areas. The only rectangle with area 1 is an  $1 \times 1$  rectangle. Similarly, we get  $1 \times 2$ ,  $1 \times 3$  rectangles for areas 2, 3 units. In the case of 4 units we may have either a  $1 \times 4$  rectangle or a  $2 \times 2$  square. Similarly, there can be a  $1 \times 5$  rectangle for area 5 units and  $1 \times 6$  or  $2 \times 3$  rectangle for 6 units. Any rectangle with area 7 units must be  $1 \times 7$  rectangle, which is not possible since the largest side could be 6 units. And any rectangle with area 8 units must be a  $2 \times 4$  rectangle. If there is any dissection of the given  $6 \times 6$  square in to 9 non-congruent rectangles with areas  $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9$ , then we observe that

$$a_1 \geq 1, a_2 \geq 2, a_3 \geq 3, a_4 \geq 4, a_5 \geq 4, a_6 \geq 5, a_7 \geq 6, a_8 \geq 6, a_9 \geq 8,$$

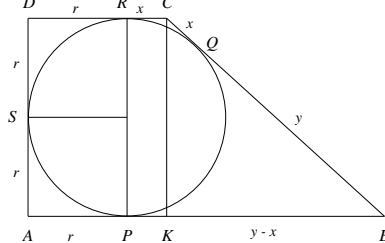
and hence the total area of all the rectangles is

$$a_1 + a_2 + \cdots + a_9 \geq 1 + 2 + 3 + 4 + 4 + 5 + 6 + 6 + 8 = 39 > 36,$$

which is the area of the given square. Hence if a  $6 \times 6$  square is dissected in to 9 rectangles as stipulated in the problem, there must be two congruent rectangles.

5. Let  $ABCD$  be a quadrilateral in which  $AB$  is parallel to  $CD$  and perpendicular to  $AD$ ;  $AB = 3CD$ ; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.

**Solution:** Let  $P, Q, R, S$  be the points of contact of in-circle with the sides  $AB, BC, CD, DA$  respectively. Since  $AD$  is perpendicular to  $AB$  and  $AB$  is parallel to  $DC$ , we see that  $AP = AS = SD = DR = r$ , the radius of the inscribed circle. Let  $BP = BQ = y$  and  $CQ = CR = x$ . Using  $AB = 3CD$ , we get  $r + y = 3(r + x)$ .



Since the area of  $ABCD$  is 4, we also get

$$4 = \frac{1}{2}AD(AB + CD) = \frac{1}{2}(2r)(4(r + x)).$$

Thus we obtain  $r(r + x) = 1$ . Using Pythagoras theorem, we obtain  $BC^2 = BK^2 + CK^2$ . However  $BC = y + x$ ,  $BK = y - x$  and  $CK = 2r$ . Substituting these and simplifying, we get  $xy = r^2$ . But  $r + y = 3(r + x)$  gives  $y = 2r + 3x$ . Thus  $r^2 = x(2r + 3x)$  and this simplifies to  $(r - 3x)(r + x) = 0$ . We conclude that  $r = 3x$ . Now the relation  $r(r + x) = 1$  implies that  $4r^2 = 3$ , giving  $r = \sqrt{3}/2$ .

6. Prove that there are infinitely many positive integers  $n$  such that  $n(n + 1)$  can be expressed as a sum of two positive squares in *at least* two different ways. (Here  $a^2 + b^2$  and  $b^2 + a^2$  are considered as the same representation.)

**Solution:** Let  $Q = n(n + 1)$ . It is convenient to choose  $n = m^2$ , for then  $Q$  is already a sum of two squares:  $Q = m^2(m^2 + 1) = (m^2)^2 + m^2$ . If further  $m^2$  itself is a sum of two squares, say  $m^2 = p^2 + q^2$ , then

$$Q = (p^2 + q^2)(m^2 + 1) = (pm + q)^2 + (p - qm)^2.$$

Note that the two representations for  $Q$  are distinct. Thus, for example, we may take  $m = 5k$ ,  $p = 3k$ ,  $q = 4k$ , where  $k$  varies over natural numbers. In this case  $n = m^2 = 25k^2$ , and

$$Q = (25k^2)^2 + (5k)^2 = (15k^2 + 4k)^2 + (20k^2 - 3k)^2.$$

As we vary  $k$  over natural numbers, we get infinitely many numbers of the form  $n(n + 1)$  each of which can be expressed as a sum of two squares in two distinct ways.

7. Let  $X$  be the set of all positive integers greater than or equal to 8 and let  $f : X \rightarrow X$  be a function such that  $f(x + y) = f(xy)$  for all  $x \geq 4$ ,  $y \geq 4$ . If  $f(8) = 9$ , determine  $f(9)$ .

**Solution:** We observe that

$$\begin{aligned} f(9) &= f(4 + 5) = f(4 \cdot 5) = f(20) = f(16 + 4) = f(16 \cdot 4) = f(64) \\ &= f(8 \cdot 8) = f(8 + 8) = f(16) = f(4 \cdot 4) = f(4 + 4) = f(8). \end{aligned}$$

Hence if  $f(8) = 9$ , then  $f(9) = 9$ . (This is one string. There may be other different ways of approaching  $f(8)$  from  $f(9)$ . The important thing to be observed is the fact that the rule  $f(x + y) = f(xy)$  applies only when  $x$  and  $y$  are at least 4. One may get strings using numbers  $x$  and  $y$  which are smaller than 4, but that is not valid. For example

$$f(9) = f(3 \cdot 3) = f(3 + 3) = f(6) = f(4 + 2) = f(4 \cdot 2) = f(8),$$

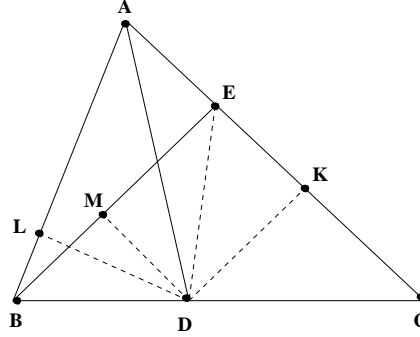
is not a valid string.)

# Solutions to CRMO-2007 Problems

- Let  $ABC$  be an acute-angled triangle;  $AD$  be the bisector of  $\angle BAC$  with  $D$  on  $BC$ ; and  $BE$  be the altitude from  $B$  on  $AC$ . Show that  $\angle CED > 45^\circ$ .

## Solution:

Draw  $DL$  perpendicular to  $AB$ ;  $DK$  perpendicular to  $AC$ ; and  $DM$  perpendicular to  $BE$ . Then  $EM = DK$ . Since  $AD$  bisects  $\angle A$ , we observe that  $\angle BAD = \angle KAD$ . Thus in triangles  $ALD$  and  $AKD$ , we see that  $\angle LAD = \angle KAD$ ;  $\angle AKD = 90^\circ = \angle ALD$ ; and  $AD$  is common. Hence triangles  $ALD$  and  $AKD$  are congruent, giving  $DL = DK$ . But  $DL > DM$ , since  $BE$  lies inside the triangle (by acuteness property). Thus  $EM > DM$ . This implies that  $\angle EDM > \angle DEM = 90^\circ - \angle EDM$ . We conclude that  $\angle EDM > 45^\circ$ . Since  $\angle CED = \angle EDM$ , the result follows.



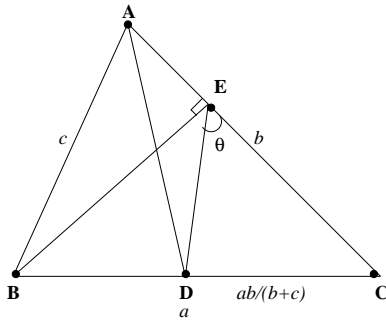
## Alternate Solution:

Let  $\angle CED = \theta$ . We have  $CD = ab/(b+c)$  and  $CE = a \cos C$ . Using sine rule in triangle  $CED$ , we have

$$\frac{CD}{\sin \theta} = \frac{CE}{\sin(C + \theta)}.$$

This reduces to

$$(b+c) \sin \theta \cos C = b \sin C \cos \theta + b \cos C \sin \theta.$$



Simplification gives  $c \sin \theta \cos C = b \sin C \cos \theta$  and so that

$$\tan \theta = \frac{b \sin C}{c \cos C} = \frac{\sin B}{\cos C} = \frac{\sin B}{\sin(\pi/2 - C)}.$$

Since  $ABC$  is acute-angled, we have  $A < \pi/2$ . Hence  $B+C > \pi/2$  or  $B > (\pi/2) - C$ . Therefore  $\sin B > \sin(\pi/2 - C)$ . This implies that  $\tan \theta > 1$  and hence  $\theta > \pi/4$ .

- Let  $a, b, c$  be three natural numbers such that  $a < b < c$  and  $\gcd(c-a, c-b) = 1$ . Suppose there exists an integer  $d$  such that  $a+d, b+d, c+d$  form the sides of a right-angled triangle. Prove that there exist integers  $l, m$  such that  $c+d = l^2 + m^2$ .

## Solution:

We have

$$(c+d)^2 = (a+d)^2 + (b+d)^2.$$

This reduces to

$$d^2 + 2d(a+b-c) + a^2 + b^2 - c^2 = 0.$$

Solving the quadratic equation for  $d$ , we obtain

$$d = -(a+b-c) \pm \sqrt{(a+b-c)^2 - (a^2 + b^2 - c^2)} = -(a+b-c) \pm \sqrt{2(c-a)(c-b)}.$$

Since  $d$  is an integer,  $2(c-a)(c-b)$  must be a perfect square; say  $2(c-a)(c-b) = x^2$ . But  $\gcd(c-a, c-b) = 1$ . Hence we have

$$c-a = 2u^2, \quad c-b = v^2 \quad \text{or} \quad c-a = u^2, \quad c-b = 2v^2,$$

where  $u > 0$  and  $v > 0$  and  $\gcd(u, v) = 1$ . In either of the cases  $d = -(a+b-c) \pm 2uv$ . In the first case

$$c+d = 2c-a-b \pm 2uv = 2u^2 + v^2 \pm 2uv = (u \pm v)^2 + u^2.$$

We observe that  $u = v$  implies that  $u = v = 1$  and hence  $c-a = 2, c-b = 1$ . Hence  $a, b, c$  are three consecutive integers. We also see that  $c+d = 1$  forcing  $b+d = 0$ , contradicting that  $b+d$  is a side of a triangle. Thus  $u \neq v$  and hence  $c+d$  is the sum of two non-zero integer squares.

Similarly, in the second case we get  $c+d = v^2 + (u \pm v)^2$ . Thus  $c+d$  is the sum of two squares.

### Alternate Solution:

One may use characterisation of primitive Pythagorean triples. Observe that  $\gcd(c-a, c-b) = 1$  implies that  $c+d, a+d, b+d$  are relatively prime. Hence there exist integers  $m > n$  such that

$$a+d = m^2 - n^2, \quad b+d = 2mn, \quad c+d = m^2 + n^2.$$

3. Find all pairs  $(a, b)$  of real numbers such that whenever  $\alpha$  is a root of  $x^2 + ax + b = 0$ ,  $\alpha^2 - 2$  is also a root of the equation.

### Solution:

Consider the equation  $x^2 + ax + b = 0$ . It has two roots (not necessarily real), say  $\alpha$  and  $\beta$ . Either  $\alpha = \beta$  or  $\alpha \neq \beta$ .

#### Case 1:

Suppose  $\alpha = \beta$ , so that  $\alpha$  is a double root. Since  $\alpha^2 - 2$  is also a root, the only possibility is  $\alpha = \alpha^2 - 2$ . This reduces to  $(\alpha+1)(\alpha-2) = 0$ . Hence  $\alpha = -1$  or  $\alpha = 2$ . Observe that  $a = -2\alpha$  and  $b = \alpha^2$ . Thus  $(a, b) = (2, 1)$  or  $(-4, 4)$ .

#### Case 2:

Suppose  $\alpha \neq \beta$ . There are four possibilities; (I)  $\alpha = \alpha^2 - 2$  and  $\beta = \beta^2 - 2$ ; (II)  $\alpha = \beta^2 - 2$  and  $\beta = \alpha^2 - 2$ ; (III)  $\alpha = \alpha^2 - 2 = \beta^2 - 2$  and  $\alpha \neq \beta$ ; or (IV)  $\beta = \alpha^2 - 2 = \beta^2 - 2$  and  $\alpha \neq \beta$

(I) Here  $(\alpha, \beta) = (2, -1)$  or  $(-1, 2)$ . Hence  $(a, b) = (-(\alpha + \beta), \alpha\beta) = (-1, -2)$ .

(II) Suppose  $\alpha = \beta^2 - 2$  and  $\beta = \alpha^2 - 2$ . Then

$$\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha).$$

Since  $\alpha \neq \beta$ , we get  $\beta + \alpha = -1$ . However, we also have

$$\alpha + \beta = \beta^2 + \alpha^2 - 4 = (\alpha + \beta)^2 - 2\alpha\beta - 4.$$

Thus  $-1 = 1 - 2\alpha\beta - 4$ , which implies that  $\alpha\beta = -1$ . Therefore  $(a, b) = (-(\alpha + \beta), \alpha\beta) = (1, -1)$ .

(III) If  $\alpha = \alpha^2 - 2 = \beta^2 - 2$  and  $\alpha \neq \beta$ , then  $\alpha = -\beta$ . Thus  $\alpha = 2, \beta = -2$  or  $\alpha = -1, \beta = 1$ . In this case  $(a, b) = (0, -4)$  and  $(0, -1)$ .

(IV) Note that  $\beta = \alpha^2 - 2 = \beta^2 - 2$  and  $\alpha \neq \beta$  is identical to (III), so that we get exactly same pairs  $(a, b)$ .

Thus we get 6 pairs;  $(a, b) = (-4, 4), (2, 1), (-1, -2), (1, -1), (0, -4), (0, -1)$ .

4. How many 6-digit numbers are there such that:

- (a) the digits of each number are all from the set  $\{1, 2, 3, 4, 5\}$ ;
- (b) any digit that appears in the number appears at least twice?

(Example: 225252 is an admissible number, while 222133 is not.)

**Solution:**

Since each digit occurs at least twice, we have following possibilities:

1. Three digits occur twice each. We may choose three digits from  $\{1, 2, 3, 4, 5\}$  in  $\binom{5}{3} = 10$  ways. If each occurs exactly twice, the number of such admissible 6-digit numbers is

$$\frac{6!}{2! 2! 2!} \times 10 = 900.$$

2. Two digits occur three times each. We can choose 2 digits in  $\binom{5}{2} = 10$  ways. Hence the number of admissible 6-digit numbers is

$$\frac{6!}{3! 3!} \times 10 = 200.$$

3. One digit occurs four times and the other twice. We are choosing two digits again, which can be done in 10 ways. The two digits are interchangeable. Hence the desired number of admissible 6-digit numbers is

$$2 \times \frac{6!}{4! 2!} \times 10 = 300.$$

4. Finally all digits are the same. There are 5 such numbers.

Thus the total number of admissible numbers is  $900 + 200 + 300 + 5 = 1405$ .

5. A trapezium  $ABCD$ , in which  $AB$  is parallel to  $CD$ , is inscribed in a circle with centre  $O$ . Suppose the diagonals  $AC$  and  $BD$  of the trapezium intersect at  $M$ , and  $OM = 2$ .

- (a) If  $\angle AMB$  is  $60^\circ$ , determine, with proof, the difference between the lengths of the parallel sides.
- (b) If  $\angle AMD$  is  $60^\circ$ , find the difference between the lengths of the parallel sides.

**Solution:**

Suppose  $\angle AMB = 60^\circ$ . Then  $AMB$  and  $CMD$  are equilateral triangles. Draw  $OK$  perpendicular to  $BD$ . (see Fig.1) Note that  $OM$  bisects  $\angle AMB$  so that  $\angle OMK =$

$30^\circ$ . Hence  $OK = OM/2 = 1$ . It follows that  $KM = \sqrt{OM^2 - OK^2} = \sqrt{3}$ . We also observe that

$$AB - CD = BM - MD = BK + KM - (DK - KM) = 2KM,$$

since  $K$  is the mid-point of  $BD$ . Hence  $AB - CD = 2\sqrt{3}$ .

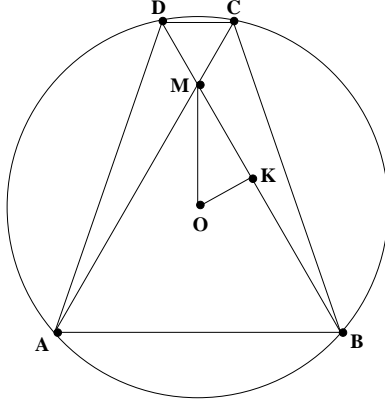


Fig. 1

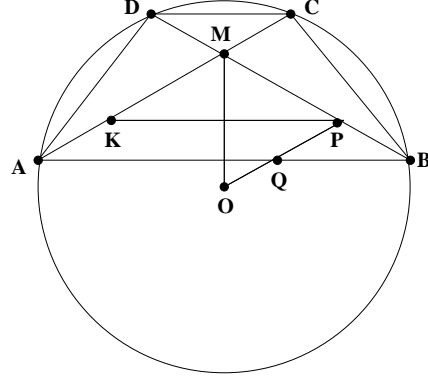


Fig. 2

Suppose  $\angle AMD = 60^\circ$  so that  $\angle AMB = 120^\circ$ . Draw  $PQ$  through  $O$  parallel to  $AC$  (with  $Q$  on  $AB$  and  $P$  on  $BD$ ). (see Fig.2) Again  $OM$  bisects  $\angle AMB$  so that  $\angle OPM = \angle OMP = 60^\circ$ . Thus  $OMP$  is an equilateral triangle. Hence diameter perpendicular to  $BD$  also bisects  $MP$ . This gives  $DM = PB$ . In the triangles  $DMC$  and  $BPQ$ , we have  $BP = DM$ ,  $\angle DMC = 120^\circ = \angle BPQ$ , and  $\angle DCM = \angle PBQ$  (property of cyclic quadrilateral). Hence  $DMC$  and  $BPQ$  are congruent so that  $DC = BQ$ . Thus  $AB - DC = AQ$ . Note that  $AQ = KP$  since  $KAQP$  is a parallelogram. But  $KP$  is twice the altitude of triangle  $OPM$ . Since  $OM = 2$ , the altitude of  $OPM$  is  $2 \times \sqrt{3}/2 = \sqrt{3}$ . This gives  $AQ = 2\sqrt{3}$ .

### Alternate Solution:

Using some trigonometry, we can get solutions for both the parts simultaneously. Let  $K, L$  be the mid-points of  $AB$  and  $CD$  respectively. Then  $L, M, O, K$  are collinear (see Fig.3 and Fig.4). Let  $\angle AMK = \theta (= \angle DML)$ , and  $OM = d$ . Since  $AMB$  and  $CMD$  are similar triangles, if  $MD = MC = x$  then  $MA = MB = kx$  for some positive constant  $k$ .

Now  $MK = kx \cos \theta$ ,  $ML = x \cos \theta$ , so that  $OK = |kx \cos \theta - d|$  and  $OL = x \cos \theta + d$ . Also  $AK = kx \sin \theta$  and  $DL = x \sin \theta$ . Using

$$AK^2 + OK^2 = AO^2 = DO^2 = DL^2 + OL^2,$$

we get

$$k^2 x^2 \sin^2 \theta + (kx \cos \theta - d)^2 = x^2 \sin^2 \theta + (x \cos \theta + d)^2.$$

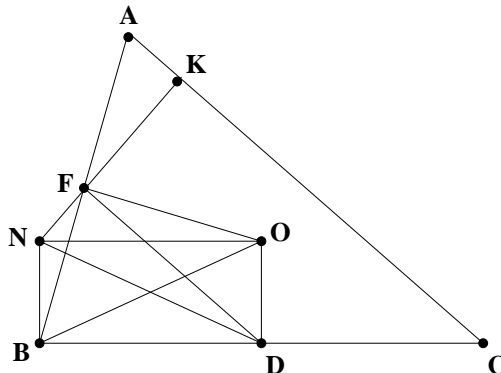


# Solutions to CRMO-2008 Problems

1. Let  $ABC$  be an acute-angled triangle; let  $D, F$  be the mid-points of  $BC, AB$  respectively. Let the perpendicular from  $F$  to  $AC$  and the perpendicular at  $B$  to  $BC$  meet in  $N$ . Prove that  $ND$  is equal to the circum-radius of  $ABC$ . [15]

**Solution:** Let  $O$  be the circum-centre of  $ABC$ . Join  $OD, ON$  and  $OF$ . We show that  $BDON$  is a rectangle. It follows that  $DN = BO = R$ , the circum-radius of  $ABC$ .

Observe that  $\angle NBC = \angle NKC = 90^\circ$ . Hence  $BCKN$  is a cyclic quadrilateral. Thus  $\angle KNB = 180^\circ - \angle BCA$ . But  $\angle BOA = 2\angle BCA$  and  $OF$  bisects  $\angle BOA$ . Hence  $\angle BOF = \angle BCA$ . We thus obtain



$$\angle FNB + \angle BOF = \angle KNB + \angle BCK = 180^\circ.$$

This implies that  $B, O, F, N$  are con-cyclic. Hence  $\angle BFO = \angle BNO$ . But observe that  $\angle BFO = 90^\circ$  since  $OF$  is perpendicular to  $AB$ . Thus  $\angle BNO = 90^\circ$ . Since  $NB$  and  $OD$  are perpendicular to  $BC$ , it follows that  $BDON$  is a rectangle.

**Alternate Solution:** We can also get the conclusion using trigonometry. Observe that  $\angle NFB = \angle AFK = 90^\circ - \angle A$ ; and  $\angle BNF = 180^\circ - \angle B$  since  $BCKN$  is a cyclic quadrilateral. Using the sine-rule in the triangle  $BFN$ ,

$$\frac{NB}{\sin \angle NFB} = \frac{BF}{\sin \angle BFN}.$$

This reduces to

$$NB = \frac{c \cos A}{2 \sin C} = R \cos A.$$

But  $BD = a/2 = R \sin A$ . Thus

$$ND^2 = NB^2 + BD^2 = R^2.$$

This gives  $ND = R$ .

2. Prove that there exist two infinite sequences  $\langle a_n \rangle_{n \geq 1}$  and  $\langle b_n \rangle_{n \geq 1}$  of positive integers such that the following conditions hold simultaneously:

- (i)  $1 < a_1 < a_2 < a_3 < \dots$ ;
- (ii)  $a_n < b_n < a_n^2$ , for all  $n \geq 1$ ;
- (iii)  $a_n - 1$  divides  $b_n - 1$ , for all  $n \geq 1$ ;
- (iv)  $a_n^2 - 1$  divides  $b_n^2 - 1$ , for all  $n \geq 1$ .

**Solution:** Let us look at the problem of finding two positive integers  $a, b$  such that  $1 < a < b < a^2$ ,  $a - 1$  divides  $b - 1$  and  $a^2 - 1$  divides  $b^2 - 1$ . Thus we have

$$b - 1 = k(a - 1), \quad \text{and} \quad b^2 - 1 = l(a^2 - 1).$$

Eliminating  $b$  from these equations, we get

$$(k^2 - l)a = k^2 - 2k + l.$$

Thus it follows that

$$a = \frac{k^2 - 2k + l}{k^2 - l} = 1 - \frac{2(k - l)}{k^2 - l}.$$

We need  $a$  to be an integer. Choose  $k^2 - l = 2$  so that  $a = 1 + l - k = k^2 - k - 1$  and  $b = k(a - 1) + 1 = k^3 - k^2 - 2k + 1$ . We want  $a > 1$  which is assured if we choose  $k \geq 3$ . Now  $a < b$  is equivalent to  $(k^2 - 1)(k - 2) > 0$  which again is assured once  $k \geq 3$ . It is easy to see that  $b < a^2$  is equivalent to  $k(k^3 - 3k^2 + 4) > 0$  and this is also true for all  $k \geq 3$ . Thus we define

$$\begin{aligned} a_n &= (n + 2)^2 - (n + 2) - 1 = n^2 + 3n + 1, \\ b_n &= (n + 2)^3 - (n + 2)^2 - 2(n + 2) + 1 = n^3 + 5n^2 + 6n + 1, \end{aligned}$$

for  $n \geq 1$ . Then we see that

$$1 < a_n < b_n < b_n^2,$$

for all  $n \geq 1$ . Moreover

$$a_n - 1 = n(n + 3), \quad b_n - 1 = n(n + 3)(n + 2)$$

and

$$a_n^2 - 1 = n(n + 3)(n + 1)(n + 2), \quad b_n^2 - 1 = n(n + 3)(n + 2)(n + 1)(n^2 + 4n + 2).$$

Thus we have a pair of desired sequences  $\langle a_n \rangle$  and  $\langle b_n \rangle$ .

3. Suppose  $a$  and  $b$  are real numbers such that the roots of the cubic equation  $ax^3 - x^2 + bx - 1 = 0$  are all positive real numbers. Prove that:

$$(i) \quad 0 < 3ab \leq 1 \quad \text{and} \quad (ii) \quad b \geq \sqrt{3}.$$

**Solution:** Let  $\alpha, \beta, \gamma$  be the roots of the given equation. We have

$$\alpha + \beta + \gamma = \frac{1}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{b}{a}, \quad \alpha\beta\gamma = \frac{1}{a}.$$

It follows that  $a, b$  are positive. We thus obtain

$$\frac{3b}{a} = 3(\alpha\beta + \beta\gamma + \gamma\alpha) \leq (\alpha + \beta + \gamma)^2 = \frac{1}{a^2},$$

which gives  $0 < 3ab \leq 1$ . Moreover

$$\begin{aligned}\frac{b^2}{a^2} &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 \\ &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \frac{2}{a^2}.\end{aligned}$$

Thus

$$\frac{b^2 - 2}{a^2} = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 \geq \frac{1}{3}(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \frac{b^2}{3a^2}.$$

This implies that  $3(b^2 - 2) \geq b^2$  or  $b^2 \geq 3$ . Hence  $b \geq \sqrt{3}$ , the conclusion follows.

4. Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0,1,2,3 occurs at least once in them. [14]

**Solution:** We observe that  $0 + 1 + 2 + 3 = 6$ . Hence the remaining two digits must account for the sum 4. This is possible with  $4 = 0 + 4 = 1 + 3 = 2 + 2$ . Thus we see that the digits in any such 6-digit number must be from one of the collections:  $\{0, 1, 2, 3, 0, 4\}$ ,  $\{0, 1, 2, 3, 1, 3\}$  or  $\{0, 1, 2, 3, 2, 2\}$ .

Consider the case in which the digits are from the collection  $\{0, 1, 2, 3, 0, 4\}$ . Here 0 occurs twice and the digits 1,2,3,4 occur once each. But 0 cannot be the first digit. Hence the first digit must be one of 1,2,3,4. Suppose we fix 1 as the first digit. Then the number of 6-digit numbers in which the remaining 5 digits are 0,0,2,3,4 is  $5!/2! = 60$ . Same is the case with other digits: 2,3,4. Thus the number of 6-digit numbers in which the digits 0,1,2,3,0,4 occur is  $60 \times 4 = 240$ .

Suppose the digits are from the collection  $\{0, 1, 2, 3, 1, 3\}$ . The number of 6-digit numbers beginning with 1 is  $5!/2! = 60$ . The number of those beginning with 2 is  $5!/(2!)(2!) = 30$  and the number of those beginning with 3 is  $5!/2! = 60$ . Thus the total number in this case is  $60 + 30 + 60 = 150$ . Alternately, we can also count it as follows: the number of 6-digit numbers one can obtain from the collection  $\{0, 1, 2, 3, 1, 3\}$  with 0 also as a possible first digit is  $6!/(2!)(2!) = 180$ ; the number of 6-digit numbers one can obtain from the collection  $\{0, 1, 2, 3, 1, 3\}$  in which 0 is the first digit is  $5!/(2!)(2!) = 30$ . Thus the number of 6-digit numbers formed by the collection  $\{0, 1, 2, 3, 1, 3\}$  such that no number has its first digit 0 is  $180 - 30 = 150$ .

Finally look at the collection  $\{0, 1, 2, 3, 2, 2\}$ . Here the number of 6-digit numbers in which 1 is the first digit is  $5!/3! = 20$ ; the number of those having 2 as the first digit is  $5!/2! = 60$ ; and the number of those having 3 as the first digit is  $5!/3! = 20$ . Thus the number of admissible 6-digit numbers here is  $20 + 60 + 20 = 100$ . This may also be obtained using the other method of counting:  $6!/3! - 5!/3! = 120 - 20 = 100$ .

Finally the total number of 6-digit numbers in which each of the digits 0,1,2,3 appears at least once is  $240 + 150 + 100 = 490$ .

5. Three nonzero real numbers  $a, b, c$  are said to be in harmonic progression if  $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ . Find all three-term harmonic progressions  $a, b, c$  of strictly increasing positive integers in which  $a = 20$  and  $b$  divides  $c$ . [17]

**Solution:** Since 20,  $b, c$  are in harmonic progression, we have

$$\frac{1}{20} + \frac{1}{c} = \frac{2}{b},$$

which reduces to  $bc + 20b - 40c = 0$ . This may also be written in the form

$$(40 - b)(c + 20) = 800.$$

Thus we must have  $20 < b < 40$  or, equivalently,  $0 < 40 - b < 20$ . Let us consider the factorisation of 800 in which one term is less than 20:

$$\begin{aligned}(40 - b)(c + 20) &= 800 = 1 \times 800 = 2 \times 400 = 4 \times 200 \\ &= 5 \times 160 = 8 \times 100 = 10 \times 80 = 16 \times 50.\end{aligned}$$

We thus get the pairs

$$(b, c) = (39, 780), (38, 380), (36, 180), (35, 140), (32, 80), (30, 60), (24, 30).$$

Among these 7 pairs, we see that only 5 pairs  $(39, 780)$ ,  $(38, 380)$ ,  $(36, 180)$ ,  $(35, 140)$ ,  $(30, 60)$  fulfill the condition of divisibility:  $b$  divides  $c$ . Thus there are 5 triples satisfying the requirement of the problem.

6. Find the number of all integer-sided *isosceles obtuse-angled* triangles with perimeter 2008. [16]

**Solution:** Let the sides be  $x, x, y$ , where  $x, y$  are positive integers. Since we are looking for obtuse-angled triangles,  $y > x$ . Moreover,  $2x + y = 2008$  shows that  $y$  is even. But  $y < x + x$ , by triangle inequality. Thus  $y < 1004$ . Thus the possible triples are  $(y, x, x) = (1002, 503, 503)$ ,  $(1000, 504, 504)$ ,  $(998, 505, 505)$ , and so on. The general form is  $(y, x, x) = (1004 - 2k, 502 + k, 502 + k)$ , where  $k = 1, 2, 3, \dots, 501$ . But the condition that the triangle is obtuse leads to

$$(1004 - 2k)^2 > 2(502 + k)^2.$$

This simplifies to

$$502^2 + k^2 - 6(502)k > 0.$$

Solving this quadratic inequality for  $k$ , we see that

$$k < 502(3 - 2\sqrt{2}), \quad \text{or} \quad k > 502(3 + 2\sqrt{2}).$$

Since  $k \leq 501$ , we can rule out the second possibility. Thus  $k < 502(3 - 2\sqrt{2})$ , which is approximately 86.1432. We conclude that  $k \leq 86$ . Thus we get 86 triangles

$$(y, x, x) = (1004 - 2k, 502 + k, 502 + k), \quad k = 1, 2, 3, \dots, 86.$$

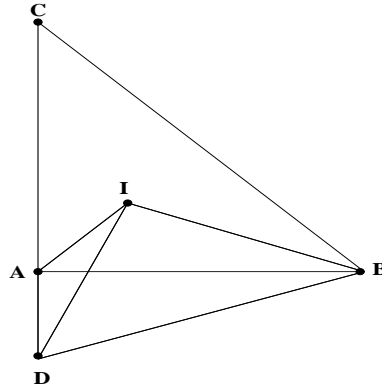
The last obtuse triangle in this list is:  $(832, 588, 588)$ . (It is easy to check that  $832^2 - 588^2 - 588^2 = 736 > 0$ , where as  $830^2 - 589^2 - 589^2 = -4942 < 0$ .)

# Regional Mathematical Olympiad-2009

## Problems and Solutions

1. Let  $ABC$  be a triangle in which  $AB = AC$  and let  $I$  be its in-centre. Suppose  $BC = AB + AI$ . Find  $\angle BAC$ .

**Solution:**



We observe that  $\angle AIB = 90^\circ + (C/2)$ . Extend  $CA$  to  $D$  such that  $AD = AI$ . Then  $CD = CB$  by the hypothesis. Hence  $\angle CDB = \angle CBD = 90^\circ - (C/2)$ . Thus

$$\angle AIB + \angle ADB = 90^\circ + (C/2) + 90^\circ - (C/2) = 180^\circ.$$

Hence  $ADBI$  is a cyclic quadrilateral. This implies that

$$\angle ADI = \angle ABI = \frac{B}{2}.$$

But  $ADI$  is isosceles, since  $AD = AI$ . This gives

$$\angle DAI = 180^\circ - 2(\angle ADI) = 180^\circ - B.$$

Thus  $\angle CAI = B$  and this gives  $A = 2B$ . Since  $C = B$ , we obtain  $4B = 180^\circ$  and hence  $B = 45^\circ$ . We thus get  $A = 2B = 90^\circ$ .

2. Show that there is no integer  $a$  such that  $a^2 - 3a - 19$  is divisible by 289.

**Solution:** We write

$$a^2 - 3a - 19 = a^2 - 3a - 70 + 51 = (a - 10)(a + 7) + 51.$$

Suppose 289 divides  $a^2 - 3a - 19$  for some integer  $a$ . Then 17 divides it and hence 17 divides  $(a - 10)(a + 7)$ . Since 17 is a prime, it must divide  $(a - 10)$  or  $(a + 7)$ . But  $(a + 7) - (a - 10) = 17$ . Hence whenever 17 divides one of  $(a - 10)$  and  $(a + 7)$ , it must divide the other also. Thus  $17^2 = 289$  divides  $(a - 10)(a + 7)$ . It follows that 289 divides 51, which is impossible. Thus, there is no integer  $a$  for which 289 divides  $a^2 - 3a - 19$ .

3. Show that  $3^{2008} + 4^{2009}$  can be written as product of two positive integers each of which is larger than  $2009^{182}$ .

**Solution:** We use the standard factorisation:

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2).$$

We observe that for any integers  $x, y$ ,

$$x^2 + 2xy + 2y^2 = (x + y)^2 + y^2 \geq y^2,$$

and

$$x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 \geq y^2.$$

We write

$$3^{2008} + 4^{2009} = 3^{2008} + 4(4^{2008}) = (3^{502})^4 + 4(4^{502})^4.$$

Taking  $x = 3^{502}$  and  $y = 4^{502}$ , we see that  $3^{2008} + 4^{2009} = ab$ , where

$$a \geq (4^{502})^2, \quad b \geq (4^{502})^2.$$

But we have

$$(4^{502})^2 = 2^{2008} > 2^{2002} = (2^{11})^{182} > (2009)^{182},$$

since  $2^{11} = 2048 > 2009$ .

4. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

**Solution:** Let  $X$  denote the set of all 3-digit natural numbers; let  $O$  be those numbers in  $X$  having only *odd* digits; and  $E$  be those numbers in  $X$  having only *even* digits. Then  $X \setminus (O \cup E)$  is the set of all 3-digit natural numbers having at least one odd digit and at least one even digit. The desired sum is therefore

$$\sum_{x \in X} x - \sum_{y \in O} y - \sum_{z \in E} z.$$

It is easy to compute the first sum;

$$\begin{aligned} \sum_{x \in X} x &= \sum_{j=1}^{999} j - \sum_{k=1}^{99} k \\ &= \frac{999 \times 1000}{2} - \frac{99 \times 100}{2} \\ &= 50 \times 9891 = 494550. \end{aligned}$$

Consider the set  $O$ . Each number in  $O$  has its digits from the set  $\{1, 3, 5, 7, 9\}$ . Suppose the digit in unit's place is 1. We can fill the digit in ten's place in 5 ways and the digit in hundred's place in 5 ways. Thus there are 25 numbers in the set  $O$  each of which has 1 in its unit's place. Similarly, there are 25 numbers whose digit in unit's place is 3; 25 having its digit in unit's place as 5; 25 with 7 and 25 with 9. Thus the sum of the digits in unit's place of all the numbers in  $O$  is

$$25(1 + 3 + 5 + 7 + 9) = 25 \times 25 = 625.$$

A similar argument shows that the sum of digits in ten's place of all the numbers in  $O$  is 625 and that in hundred's place is also 625. Thus the sum of all the numbers in  $O$  is

$$625(10^2 + 10 + 1) = 625 \times 111 = 69375.$$

Consider the set  $E$ . The digits of numbers in  $E$  are from the set  $\{0, 2, 4, 6, 8\}$ , but the digit in hundred's place is never 0. Suppose the digit in unit's place is 0. There are  $4 \times 5 = 20$  such numbers. Similarly, 20 numbers each having digits 2,4,6,8 in their unit's place. Thus the sum of the digits in unit's place of all the numbers in  $E$  is

$$20(0 + 2 + 4 + 6 + 8) = 20 \times 20 = 400.$$

A similar reasoning shows that the sum of the digits in ten's place of all the numbers in  $E$  is 400, but the sum of the digits in hundred's place of all the numbers in  $E$  is  $25 \times 20 = 500$ . Thus the sum of all the numbers in  $E$  is

$$500 \times 10^2 + 400 \times 10 + 400 = 54400.$$

The required sum is

$$494550 - 69375 - 54400 = 370775.$$

5. A convex polygon  $\Gamma$  is such that the distance between any two vertices of  $\Gamma$  does not exceed 1.

- (i) Prove that the distance between any two points on the boundary of  $\Gamma$  does not exceed 1.
- (ii) If  $X$  and  $Y$  are two distinct points inside  $\Gamma$ , prove that there exists a point  $Z$  on the boundary of  $\Gamma$  such that  $XZ + YZ \leq 1$ .

**Solution:**

- (i) Let  $S$  and  $T$  be two points on the boundary of  $\Gamma$ , with  $S$  lying on the side  $AB$  and  $T$  lying on the side  $PQ$  of  $\Gamma$ . (See Fig. 1.) Join  $TA, TB, TS$ . Now  $ST$  lies between  $TA$  and  $TB$  in triangle  $TAB$ . One of  $\angle AST$  and  $\angle BST$  is at least  $90^\circ$ , say  $\angle AST \geq 90^\circ$ . Hence  $AT \geq TS$ . But  $AT$  lies inside triangle  $APQ$  and one of  $\angle ATP$  and  $\angle ATQ$  is at least  $90^\circ$ , say  $\angle ATP \geq 90^\circ$ . Then  $AP \geq AT$ . Thus we get  $TS \leq AT \leq AP \leq 1$ .

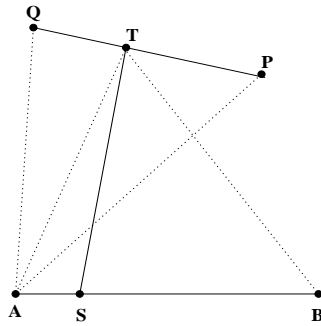


Fig. 1

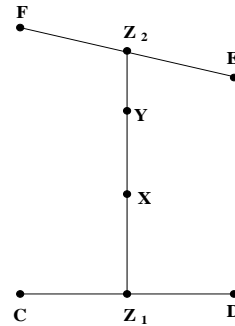


Fig. 2

- (ii) Let  $X$  and  $Y$  be points in the interior  $\Gamma$ . Join  $XY$  and produce them on either side to meet the sides  $CD$  and  $EF$  of  $\Gamma$  at  $Z_1$  and  $Z_2$  respectively. We have

$$\begin{aligned}(XZ_1 + YZ_1) + (XZ_2 + YZ_2) &= (XZ_1 + XZ_2) + (YZ_1 + YZ_2) \\ &= 2Z_1Z_2 \leq 2,\end{aligned}$$

by the first part. Therefore one of the sums  $XZ_1 + YZ_1$  and  $XZ_2 + YZ_2$  is at most 1. We may choose  $Z$  accordingly as  $Z_1$  or  $Z_2$ .

6. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

**Solution:** Suppose  $r$  pages of the book are torn off. Note that the page numbers on both the sides of a page are of the form  $2k - 1$  and  $2k$ , and their sum is  $4k - 1$ . The sum of the numbers on the torn pages must be of the form

$$4k_1 - 1 + 4k_2 - 1 + \cdots + 4k_r - 1 = 4(k_1 + k_2 + \cdots + k_r) - r.$$

The sum of the numbers of all the pages in the untorn book is

$$1 + 2 + 3 + \cdots + 100 = 5050.$$

Hence the sum of the numbers on the torn pages is

$$5050 - 4949 = 101.$$

We therefore have

$$4(k_1 + k_2 + \cdots + k_r) - r = 101.$$

This shows that  $r \equiv 3 \pmod{4}$ . Thus  $r = 4l + 3$  for some  $l \geq 0$ .

Suppose  $r \geq 7$ , and suppose  $k_1 < k_2 < k_3 < \cdots < k_r$ . Then we see that

$$\begin{aligned}4(k_1 + k_2 + \cdots + k_r) - r &\geq 4(k_1 + k_2 + \cdots + k_7) - 7 \\ &\geq 4(1 + 2 + \cdots + 7) - 7 \\ &= 4 \times 28 - 7 = 105 > 101.\end{aligned}$$

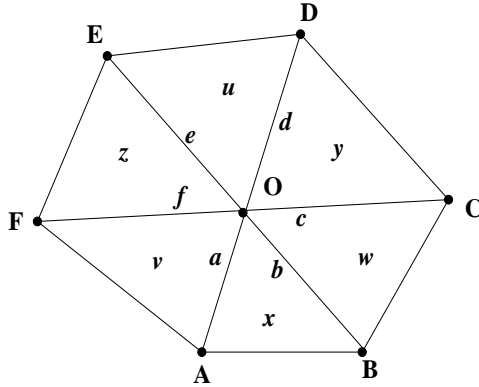
Hence  $r = 3$ . This leads to  $k_1 + k_2 + k_3 = 26$  and one can choose distinct positive integers  $k_1, k_2, k_3$  in several ways.

—————00—————

# Regional Mathematical Olympiad-2010

## Problems and Solutions

1. Let  $ABCDEF$  be a convex hexagon in which the diagonals  $AD$ ,  $BE$ ,  $CF$  are concurrent at  $O$ . Suppose the area of triangle  $OAF$  is the geometric mean of those of  $OAB$  and  $OEF$ ; and the area of triangle  $OBC$  is the geometric mean of those of  $OAB$  and  $OCD$ . Prove that the area of triangle  $OED$  is the geometric mean of those of  $OCD$  and  $OEF$ .



**Solution:** Let  $OA = a$ ,  $OB = b$ ,  $OC = c$ ,  $OD = d$ ,  $OE = e$ ,  $OF = f$ ,  $[OAB] = x$ ,  $[OCD] = y$ ,  $[OEF] = z$ ,  $[ODE] = u$ ,  $[OFA] = v$  and  $[OBC] = w$ . We are given that  $v^2 = zx$ ,  $w^2 = xy$  and we have to prove that  $u^2 = yz$ .

Since  $\angle AOB = \angle DOE$ , we have

$$\frac{u}{x} = \frac{\frac{1}{2}de \sin \angle DOE}{\frac{1}{2}ab \sin \angle AOB} = \frac{de}{ab}.$$

Similarly, we obtain

$$\frac{v}{y} = \frac{fa}{cd}, \quad \frac{w}{z} = \frac{bc}{ef}.$$

Multiplying, these three equalities, we get  $uvw = xyz$ . Hence

$$x^2 y^2 z^2 = u^2 v^2 w^2 = u^2 (zx)(xy).$$

This gives  $u^2 = yz$ , as desired.

2. Let  $P_1(x) = ax^2 - bx - c$ ,  $P_2(x) = bx^2 - cx - a$ ,  $P_3(x) = cx^2 - ax - b$  be three quadratic polynomials where  $a, b, c$  are non-zero real numbers. Suppose there exists a real number  $\alpha$  such that  $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$ . Prove that  $a = b = c$ .

**Solution:** We have three relations:

$$\begin{aligned} a\alpha^2 - b\alpha - c &= \lambda, \\ b\alpha^2 - c\alpha - a &= \lambda, \\ c\alpha^2 - a\alpha - b &= \lambda, \end{aligned}$$

where  $\lambda$  is the common value. Eliminating  $\alpha^2$  from these, taking these equations pairwise, we get three relations:

$$\begin{aligned} (ca - b^2)\alpha - (bc - a^2) &= \lambda(b - a), & (ab - c^2)\alpha - (ca - b^2) &= \lambda(c - b), \\ (bc - a^2) - (ab - c^2) &= \lambda(a - c). \end{aligned}$$

Adding these three, we get

$$(ab + bc + ca - a^2 - b^2 - c^2)(\alpha - 1) = 0.$$

(Alternatively, multiplying above relations respectively by  $b - c$ ,  $c - a$  and  $a - b$ , and adding also leads to this.) Thus either  $ab + bc + ca - a^2 - b^2 - c^2 = 0$  or  $\alpha = 1$ . In the first case

$$0 = ab + bc + ca - a^2 - b^2 - c^2 = \frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2)$$

shows that  $a = b = c$ . If  $\alpha = 1$ , then we obtain

$$a - b - c = b - c - a = c - a - b,$$

and once again we obtain  $a = b = c$ .

3. Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

**Solution:** We divide the even 4-digit numbers having non-zero digits into 4 classes: those ending in 2, 4, 6, 8.

- (A) Suppose a 4-digit number ends in 2. Then the second right digit must be odd in order to be divisible by 4. Thus the last 2 digits must be of the form 12, 32, 52, 72 or 92. If a number ends in 12, 52 or 92, then the previous digit must be even in order *not* to be divisible by 8 and we have 4 admissible even digits. Now the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get  $9 \times 4 \times 3 = 108$  such numbers. If a number ends in 32 or 72, then the previous digit must be odd in order *not* to be divisible by 8 and we have 5 admissible odd digits. Here again the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get  $9 \times 5 \times 2 = 90$  such numbers. Thus the number of 4-digit numbers having non-zero digits, ending in 2, divisible by 4 but not by 8 is  $108 + 90 = 198$ .
- (B) If the number ends in 4, then the previous digit must be even for divisibility by 4. Thus the last two digits must be of the form 24, 44, 54, 84. If we take numbers ending with 24 and 64, then the previous digit must be odd for non-divisibility by 8 and the left most digit can be any non-zero digit. Here we get  $9 \times 5 \times 2 = 90$  such numbers. If the last two digits are of the form 44 and 84, then previous digit must be even for non-divisibility by 8. And the left most digit can take 9 possible values. We thus get  $9 \times 4 \times 2 = 72$  numbers. Thus the admissible numbers ending in 4 is  $90 + 72 = 162$ .
- (C) If a number ends with 6, then the last two digits must be of the form 16, 36, 56, 76, 96. For numbers ending with 16, 56, 76, the previous digit must be odd. For numbers ending with 36, 76, the previous digit must be even. Thus we get here  $(9 \times 5 \times 3) + (9 \times 4 \times 2) = 135 + 72 = 207$  numbers.
- (D) If a number ends with 8, then the last two digits must be of the form 28, 48, 68, 88. For numbers ending with 28, 68, the previous digit must be even. For numbers ending with 48, 88, the previous digit must be odd. Thus we get  $(9 \times 4 \times 2) + (9 \times 5 \times 2) = 72 + 90 = 162$  numbers.

Thus the number of 4-digit numbers, having non-zero digits, and divisible by 4 but not by 8 is

$$198 + 162 + 207 + 162 = 729.$$

**Alternative Solution:.** If we take any four consecutive even numbers and divide them by 8, we get remainders 0, 2, 4, 6 in some order. Thus there is only one number of the form  $8k + 4$  among them which is divisible by 4 but not by 8. Hence if we take four even consecutive numbers

$$\begin{aligned} 1000a + 100b + 10c + 2, \quad 1000a + 100b + 10c + 4, \\ 1000a + 100b + 10c + 6, \quad 1000a + 100b + 10c + 8, \end{aligned}$$

there is exactly one among these four which is divisible by 4 but not by 8. Now we can divide the set of all 4-digit even numbers with non-zero digits into groups of 4 such

consecutive even numbers with  $a, b, c$  nonzero. And in each group, there is exactly one number which is divisible by 4 but not by 8. The number of such groups is precisely equal to  $9 \times 9 \times 9 = 729$ , since we can vary  $a, b, c$  in the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ .

4. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

**Solution:** Let  $x, y, z$  be three distinct positive integers satisfying the given conditions. We may assume that  $x < y < z$ . Thus we have three relations:

$$\frac{1}{y} + \frac{1}{z} = \frac{a}{x}, \quad \frac{1}{z} + \frac{1}{x} = \frac{b}{y}, \quad \frac{1}{x} + \frac{1}{y} = \frac{c}{z},$$

for some positive integers  $a, b, c$ . Thus

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a+1}{x} = \frac{b+1}{y} = \frac{c+1}{z} = r,$$

say. Since  $x < y < z$ , we observe that  $a < b < c$ . We also get

$$\frac{1}{x} = \frac{r}{a+1}, \quad \frac{1}{y} = \frac{r}{b+1}, \quad \frac{1}{z} = \frac{r}{c+1}.$$

Adding these, we obtain

$$r = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{r}{a+1} + \frac{r}{b+1} + \frac{r}{c+1},$$

or

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1. \tag{1}$$

Using  $a < b < c$ , we get

$$1 = \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} < \frac{3}{a+1}.$$

Thus  $a < 2$ . We conclude that  $a = 1$ . Putting this in the relation (1), we get

$$\frac{1}{b+1} + \frac{1}{c+1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence  $b < c$  gives

$$\frac{1}{2} < \frac{2}{b+1}.$$

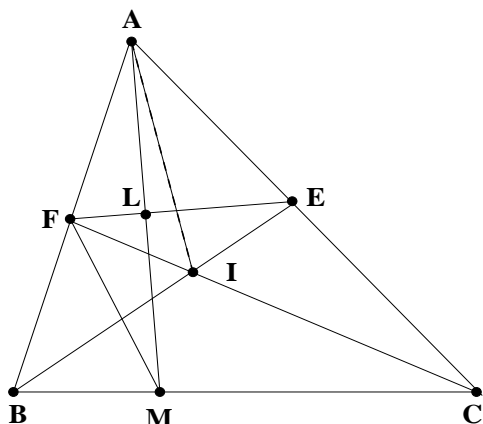
Thus  $b+1 < 4$  or  $b < 3$ . Since  $b > a = 1$ , we must have  $b = 2$ . This gives

$$\frac{1}{c+1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

or  $c = 5$ . Thus  $x : y : z = a+1 : b+1 : c+1 = 2 : 3 : 6$ . Thus the required numbers with the least sum are 2, 3, 6.

**Alternative Solution:** We first observe that  $(1, a, b)$  is not a solution whenever  $1 < a < b$ . Otherwise we should have  $\frac{1}{a} + \frac{1}{b} = \frac{l}{1} = l$  for some integer  $l$ . Hence we obtain  $\frac{a+b}{ab} = l$  showing that  $a|b$  and  $b|a$ . Thus  $a = b$  contradicting  $a \neq b$ . Thus the least number should be 2. It is easy to verify that  $(2, 3, 4)$  and  $(2, 3, 5)$  are not solutions and  $(2, 3, 6)$  satisfies all the conditions. (We may observe  $(2, 4, 5)$  is also not a solution.) Since  $3 + 4 + 5 = 12 > 11 = 2 + 3 + 6$ , it follows that  $(2, 3, 6)$  has the required minimality.

5. Let  $ABC$  be a triangle in which  $\angle A = 60^\circ$ . Let  $BE$  and  $CF$  be the bisectors of the angles  $\angle B$  and  $\angle C$  with  $E$  on  $AC$  and  $F$  on  $AB$ . Let  $M$  be the reflection of  $A$  in the line  $EF$ . Prove that  $M$  lies on  $BC$ .



**Solution:** Draw  $AL \perp EF$  and extend it to meet  $BC$  in  $M$ . We show that  $AL = LM$ . First we show that  $A, F, I, E$  are concyclic. We have

$$\angle BIC = 90^\circ + \frac{\angle A}{2} = 90^\circ + 30^\circ = 120^\circ.$$

Hence  $\angle FIE = \angle BIC = 120^\circ$ . Since  $\angle A = 60^\circ$ , it follows that  $A, F, I, E$  are concyclic. Hence  $\angle BEF = \angle IEF = \angle IAF = \angle A/2$ . This gives

$$\angle AFE = \angle ABE + \angle BEF = \frac{\angle B}{2} + \frac{\angle A}{2}.$$

Since  $\angle ALF = 90^\circ$ , we see that

$$\angle FAM = 90^\circ - \angle AFE = 90^\circ - \frac{\angle B}{2} - \frac{\angle A}{2} = \frac{\angle C}{2} = \angle FCM.$$

This implies that  $F, M, C, A$  are concyclic. It follows that

$$\angle FMA = \angle FCA = \frac{\angle C}{2} = \angle FAM.$$

Hence  $FMA$  is an isosceles triangle. But  $FL \perp AM$ . Hence  $L$  is the mid-point of  $AM$  or  $AL = LM$ .

6. For each integer  $n \geq 1$ , define  $a_n = \left[ \frac{n}{\lfloor \sqrt{n} \rfloor} \right]$ , where  $[x]$  denotes the largest integer not exceeding  $x$ , for any real number  $x$ . Find the number of all  $n$  in the set  $\{1, 2, 3, \dots, 2010\}$  for which  $a_n > a_{n+1}$ .

**Solution:** Let us examine the first few natural numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9. Here we see that  $a_n = 1, 2, 3, 2, 2, 3, 3, 4, 3$ . We observe that  $a_n \leq a_{n+1}$  for all  $n$  except when  $n + 1$  is a square in which case  $a_n > a_{n+1}$ . We prove that this observation is valid in general. Consider the range

$$m^2, m^2 + 1, m^2 + 2, \dots, m^2 + m, m^2 + m + 1, \dots, m^2 + 2m.$$

Let  $n$  take values in this range so that  $n = m^2 + r$ , where  $0 \leq r \leq 2m$ . Then we see that  $\lfloor \sqrt{n} \rfloor = m$  and hence

$$\left[ \frac{n}{\lfloor \sqrt{n} \rfloor} \right] = \left[ \frac{m^2 + r}{m} \right] = m + \left[ \frac{r}{m} \right].$$

Thus  $a_n$  takes the values  $\underbrace{m, m, m, \dots, m}_{m \text{ times}}, \underbrace{m+1, m+1, m+1, \dots, m+1}_{m \text{ times}}, m+2$ , in this

range. But when  $n = (m+1)^2$ , we see that  $a_n = m+1$ . This shows that  $a_{n-1} > a_n$  whenever  $n = (m+1)^2$ . When we take  $n$  in the set  $\{1, 2, 3, \dots, 2010\}$ , we see that the only squares are  $1^2, 2^2, \dots, 44^2$  (since  $44^2 = 1936$  and  $45^2 = 2025$ ) and  $n = (m+1)^2$  is possible for only 43 values of  $m$ . Thus  $a_n > a_{n+1}$  for 43 values of  $n$ . (These are  $2^2 - 1, 3^2 - 1, \dots, 44^2 - 1$ .)

### Model solutions of RMO 2012 (Mumbai region)

1. Let  $\alpha$  be the common zero of the three given polynomials. Then

$$\alpha^2 + a\alpha + b = 0; \quad (1)$$

$$\alpha^2 + \alpha + ab = 0; \quad (2)$$

$$a\alpha^2 + \alpha + b = 0. \quad (3)$$

(1)-(2) yields

$$(a - 1)(\alpha - b) = 0. \quad (4)$$

(3)-(2) yields

$$(a - 1)(\alpha^2 - b) = 0. \quad (5)$$

From (4) we conclude that  $a = 1$  or  $\alpha = b$  and from (5) we see that  $a = 1$  or  $\alpha^2 = b$ . But if  $a = 1$  then the three polynomials are same as  $x^2 + x + b$ , contradicting the fact that they are different. Therefore we must have  $\alpha = b$  and  $\alpha^2 = b$ . Thus  $\alpha = \alpha^2$  i.e  $\alpha = 0$  or  $\alpha = 1$ . If  $\alpha = 0$  then  $b = 0$  and this is not a feasible solution since we need to find  $b \neq 0$ . Hence  $\alpha = 1$ , which yields  $b = 1$  and from (1),  $a = -2$ .

Thus  $a = -2$ ,  $b = 1$  and the polynomials are  $x^2 - 2x + 1$ ,  $x^2 + x - 2$  and  $-2x^2 + x + 1$ .

2. Observe that  $n^2 + 3n + 51 = (n - 5)(n + 8) + 91$ . If  $13|n^2 + 3n + 51$ , then  $13|(n - 5)(n + 8)$ . Therefore  $13|n - 5$  or  $13|n + 8$ . Observe that

$$13|n - 5 \Leftrightarrow 13|(n + 8) - 13 \Leftrightarrow 13|n + 8;$$

Now,  $21n^2 + 89n + 44 = (7n + 4)(3n + 11) = \{(3(n + 8) + 4(n - 5))\}\{2(n + 8) + (n - 5)\}$ . Writing  $n + 8 = 13m_1$  and  $n - 5 = 13m_2$ , where  $m_1$  and  $m_2$  are positive integers we get

$$21n^2 + 89n + 44 = 169(3m_1 + 4m_2)(2m_1 + m_2).$$

Therefore 169 divides  $21n^2 + 89n + 44$ .

#### Comments:

There were different solutions to this problem by the students. We present two such solutions.

#### First solution:

Observe that  $n^2 + 3n + 51 \equiv 0 \equiv \{(n - 5)^2 + 26\} \pmod{13} \Rightarrow n \equiv 5 \pmod{13}$ . Now  $21n^2 + 89n + 44 = (7n + 4)(3n + 11) \equiv 0 \pmod{169}$  because  $7n + 4 \equiv 39 \equiv 0 \pmod{13}$  and  $3n + 11 \equiv 26 \equiv 0 \pmod{13}$ .

#### Second solution:

$13|n^2 + 3n + 51 \Rightarrow 13|(n + 8)(n - 5) \Rightarrow n \equiv 5 \pmod{13}$ . Let  $n_k = 13k + 5$ ,  $k$  an integer, and  $f(n_k) = 21n_k^2 + 89n_k + 44$ . Then

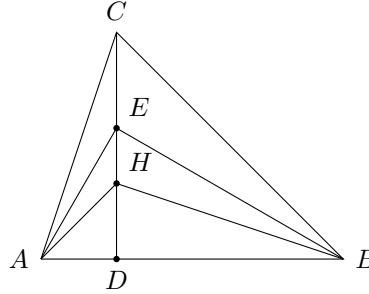
$$f(n_{k+1}) - f(n_k) = 169(44 + 42k) \equiv 0 \pmod{169} \Rightarrow f(n_k) \equiv f(n_0) \pmod{169}.$$

As  $f(n_0) = 1014 \equiv 0 \pmod{169}$  we conclude that  $f(n_k) \equiv 0 \pmod{169}$ .

3. Writing  $x = [x] + \{x\}$  in the given equation and simplifying we obtain  $2^{5\{x\}} = \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11}$ .  
 As  $0 \leq \{x\} < 1$ ,  $1 \leq 2^{5\{x\}} < 32$  and hence the right hand side is positive. Therefore  $2^{2[x]} - 11 > 0$ , i.e  $[x] \geq 2$ . Also,  $5 < \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11} \leq 16$ . Hence there is a solution for every real  $x$  with  $[x] \geq 2$  of which only one is rational, namely  $x = 14/5$ .
4. Since  $\angle AEB = 90^\circ$  and  $ED$  is perpendicular to  $AB$ ,  $ED^2 = AD \cdot DB$ . Now

$$[ABE]^2 = [ABC][ABH] \Leftrightarrow ED^2 = HD \cdot CD \Leftrightarrow AD \cdot DB = HD \cdot CD.$$

But triangles  $ADH$  and  $CDB$  are similar because  $\angle DAH = \angle BCD$ ,  $\angle ADH = \angle CDB$ .



Therefore,  $AD/CD = HD/DB$ , i.e  $AD \cdot DB = HD \cdot CD$ . Thus  $[ABE]^2 = [ABC][ABH]$ .

5. We have  $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$  and  $1 \leq a \leq b \leq c$ . Therefore  $1/a \geq 1/b \geq 1/c$  and

$$1 = \frac{1}{a} + \frac{2}{b} + \frac{3}{c} \leq \frac{1}{a} + \frac{2}{a} + \frac{3}{a} = \frac{6}{a} \Rightarrow a \leq 6.$$

Since  $a$  is a prime and  $1 \leq a \leq 6$ , the possible values of  $a$  are 2, 3 and 5.

**Case 1:**  $a = 2$ .

$$a = 2 \Rightarrow \frac{2}{b} + \frac{3}{c} = \frac{1}{2}. \text{ Now } \frac{2}{b} < \frac{1}{2} \Rightarrow b \geq 5 \text{ and } 1/b \geq 1/c \Rightarrow \frac{1}{2} = \frac{2}{b} + \frac{3}{c} \leq \frac{5}{b} \Rightarrow b \leq 10.$$

Hence  $5 \leq b \leq 10$ . Substituting the possible values of  $b$  in the equation  $\frac{2}{b} + \frac{3}{c} = \frac{1}{2}$  we obtain  $(b, c) = (5, 30), (6, 18), (7, 14), (8, 12), (10, 10)$  as the admissible pairs. Therefore the solutions in this case are  $(a, b, c) = (2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10)$ .

**Case 2:**  $a = 3$ .

Emulating the method outlined in the analysis of **Case 1** we find that  $4 \leq b \leq 7$  and the solutions in this case are  $(a, b, c) = (3, 4, 18), (3, 6, 9)$ .

**Case 3:**  $a = 5$ .

There is no solution in this case.

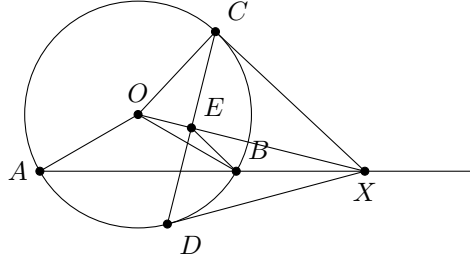
In summary, the solutions are  $(a, b, c) = (2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10), (3, 4, 18), (3, 6, 9)$ .

6.  $WSUM = 70656$ . Partition the set of all subsets of  $S$  into two sets - $T_1$ - consisting of those subsets which contain the element 1 and  $T_2$  those which do not contain 1. For every subset  $S_2$  belonging to  $T_2$ , there is a unique subset  $S_1$  belonging to  $T_1$  which is  $\{1\} \cup S_2$ . Consider any element  $a \geq 2$  of  $S$ . Let  $S_2$  be such that the element  $a$  occupies an even numbered position in it. It will occupy an odd numbered position in  $S_1$ . The total contribution to  $WSUM$  by  $a$  from both these subsets is  $5a$ . The same is true if  $a$  occupies an odd numbered position in  $S_2$ . Therefore, the total contribution of the element  $a$  to  $WSUM$  is  $5a$  multiplied by the number of subsets of  $T_2$  that contain  $a$ . The number of these subsets is  $2^8$ . The contribution of the element 1 to  $WSUM$  is clearly  $3 \times 2^9$ . Therefore the sum of all  $WSUM$ s is

$$T = 3 \times 2^9 + 5 \times 2^8 \times \sum_{j=2}^{10} j = 276 \times 2^8 = 70656.$$

Aliter: The total contribution of the element  $a \geq 2$  over all subsets is  $t = \sum_{j=0}^{a-1} \binom{a-1}{j} (2 + \xi(j)) 2^{10-a} a = 0$ . Here  $\xi(j) = 0$  if  $j$  is odd and  $\xi(j) = 1$  if  $j$  is even.  $\binom{a-1}{j}$  is the number of  $j$ -element subsets of  $\{1, 2, \dots, a-1\}$  and  $2^{10-a}$  is the total number of subsets of  $\{a+1, a+2, \dots, 10\}$ . Simplifying, we get  $t = 2 \cdot 2^9 a + 2^8 a = 5 \cdot 2^8 a$ .

7.  $O$ ,  $E$ , and  $X$  are collinear. Join  $O$  with  $A$ ,  $B$  and  $C$ . Triangles  $OCX$  and  $CEX$  are similar.



Therefore  $XC/XO = XE/XC$ , i.e.  $XC^2 = XO \cdot XE$ . But  $XC^2 = XB \cdot XA$ . Hence  $XB \cdot XA = XE \cdot XO$  implying  $B, A, O, E$  are concyclic. Therefore  $\angle OAB = 180^\circ - \angle OEB = 40^\circ$ . So,  $\angle AOB = 180^\circ - 2\angle OAB = 100^\circ$ .

8. Put  $x = 2a$ ,  $y = 2b$  and  $z = 2c$ . The problem reduces to showing

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}$$

subject to  $1/a + 1/b + 1/c = 1$ . Observe that  $a > 1, b > 1, c > 1$  and  $a-1 = a \left( \frac{1}{b} + \frac{1}{c} \right) \geq \frac{2a}{\sqrt{bc}}$  (by A.M-G.M inequality). Similarly we get  $b-1 \geq \frac{2b}{\sqrt{ca}}$  and  $c-1 \geq \frac{2c}{\sqrt{ab}}$ . Multiplying these and taking the reciprocal we obtain

$$\frac{1}{(a-1)(b-1)(c-1)} \leq \frac{1}{8}. \dots (I)$$

Next observe that  $\frac{a+1}{a-1} = 1 + \frac{2}{a-1} \geq \frac{2\sqrt{2}}{\sqrt{a-1}}$  whence  $a+1 \geq 2\sqrt{2(a-1)}$ . Similarly we obtain  $b+1 \geq 2\sqrt{2(b-1)}$  and  $c+1 \geq 2\sqrt{2(c-1)}$ . Multiplying these yields

$$(a+1)(b+1)(c+1) \geq 16\sqrt{2(a-1)(b-1)(c-1)} \geq 16\sqrt{2 \cdot 8} = 64.$$

Therefore

$$\frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{8} \cdot \dots \text{ (II)}$$

By adding (I) and (II) we get

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

**Comments:**

We present another method which many students had adopted. By the A.M-G.M-H.M inequality,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{1/a+1/b+1/c} = 3.$$

Thus  $a+b+c \geq 9$  and  $abc \geq 27$ . Using these two inequalities we obtain

$$\begin{aligned} (a-1)(b-1)(c-1) &= abc - (ab+bc+ca) + (a+b+c) - 1 \geq 8, \\ (a+1)(b+1)(c+1) &= abc + (ab+bc+ca) + (a+b+c) + 1 = 2abc + (a+b+c) + 1 \geq 2(27) + 9 + 1 = 64. \end{aligned}$$

From these two inequalities we get

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

1. Let  $ABC$  be an isosceles triangle with  $AB = AC$  and let  $\Gamma$  denote its circumcircle. A point  $D$  is on arc  $AB$  of  $\Gamma$  not containing  $C$ . A point  $E$  is on arc  $AC$  of  $\Gamma$  not containing  $B$ . If  $AD = CE$  prove that  $BE$  is parallel to  $AD$ .
2. Find all triples  $(p, q, r)$  of primes such that  $pq = r + 1$  and  $2(p^2 + q^2) = r^2 + 1$ .
3. A finite non-empty set of integers is called *3-good* if the sum of its elements is divisible by 3. Find the number of non-empty 3-good subsets of  $\{0, 1, 2, \dots, 9\}$ .
4. In a triangle  $ABC$ , points  $D$  and  $E$  are on segments  $BC$  and  $AC$  such that  $BD = 3DC$  and  $AE = 4EC$ . Point  $P$  is on line  $ED$  such that  $D$  is the midpoint of segment  $EP$ . Lines  $AP$  and  $BC$  intersect at point  $S$ . Find the ratio  $BS/SD$ .
5. Let  $a_1, b_1, c_1$  be natural numbers. We define

$$a_2 = \gcd(b_1, c_1), \quad b_2 = \gcd(c_1, a_1), \quad c_2 = \gcd(a_1, b_1),$$

and

$$a_3 = \text{lcm}(b_2, c_2), \quad b_3 = \text{lcm}(c_2, a_2), \quad c_3 = \text{lcm}(a_2, b_2).$$

Show that  $\gcd(b_3, c_3) = a_2$ .

6. Let  $P(x) = x^3 + ax^2 + b$  and  $Q(x) = x^3 + bx + a$ , where  $a, b$  are non-zero real numbers. Suppose that the roots of the equation  $P(x) = 0$  are the reciprocals of the roots of the equation  $Q(x) = 0$ . Prove that  $a$  and  $b$  are integers. Find the greatest common divisor of  $P(2013! + 1)$  and  $Q(2013! + 1)$ .

1. Let  $ABC$  be an isosceles triangle with  $AB = AC$  and let  $\Gamma$  denote its circumcircle. A point  $D$  is on the arc  $AB$  of  $\Gamma$  not containing  $C$  and a point  $E$  is on the arc  $AC$  of  $\Gamma$  not containing  $B$  such that  $AD = CE$ . Prove that  $BE$  is parallel to  $AD$ .

**Solution.** We note that triangle  $AEC$  and triangle  $BDA$  are congruent. Therefore  $AE = BD$  and hence  $\angle ABE = \angle DAB$ . This proves that  $AD$  is parallel to  $BE$ .  $\square$

2. Find all triples  $(p, q, r)$  of primes such that  $pq = r + 1$  and  $2(p^2 + q^2) = r^2 + 1$ .

**Solution.** If  $p$  and  $q$  are both odd, then  $r = pq - 1$  is even so  $r = 2$ . But in this case  $pq \geq 3 \times 3 = 9$  and hence there are no solutions. This proves that either  $p = 2$  or  $q = 2$ . If  $p = 2$  then we have  $2q = r + 1$  and  $8 + 2q^2 = r^2 + 1$ . Multiplying the second equation by 2 we get  $2r^2 + 2 = 16 + (2q)^2 = 16 + (r + 1)^2$ . Rearranging the terms, we have  $r^2 - 2r - 15 = 0$ , or equivalently  $(r + 3)(r - 5) = 0$ . This proves that  $r = 5$  and hence  $q = 3$ . Similarly, if  $q = 2$  then  $r = 5$  and  $p = 3$ . Thus the only two solutions are  $(p, q, r) = (2, 3, 5)$  and  $(p, q, r) = (3, 2, 5)$ .  $\square$

3. A finite non-empty set  $S$  of integers is called 3-good if the sum of the elements of  $S$  is divisible by 3. Find the number of 3-good non-empty subsets of  $\{0, 1, 2, \dots, 9\}$ .

**Solution.** Let  $A$  be a 3-good subset of  $\{0, 1, \dots, 9\}$ . Let  $A_1 = A \cap \{0, 3, 6, 9\}$ ,  $A_2 = A \cap \{1, 4, 7\}$  and  $A_3 = A \cap \{2, 5, 8\}$ . Then there are three possibilities:

- $|A_2| = 3, |A_3| = 0$ ;
- $|A_2| = 0, |A_3| = 3$ ;
- $|A_2| = |A_3|$ .

Note that there are 16 possibilities for  $A_1$ . Therefore the first two cases correspond to a total of 32 subsets that are 3-good. The number of subsets in the last case is  $16(1^2 + 3^2 + 3^2 + 1^2) = 320$ . Note that this also includes the empty set. Therefore there are a total of 351 non-empty 3-good subsets of  $\{0, 1, 2, \dots, 9\}$ .  $\square$

4. In a triangle  $ABC$ , points  $D$  and  $E$  are on segments  $BC$  and  $AC$  such that  $BD = 3DC$  and  $AE = 4EC$ . Point  $P$  is on line  $ED$  such that  $D$  is the midpoint of segment  $EP$ . Lines  $AP$  and  $BC$  intersect at point  $S$ . Find the ratio  $BS/SD$ .

**Solution.** Let  $F$  denote the midpoint of the segment  $AE$ . Then it follows that  $DF$  is parallel to  $AP$ . Therefore, in triangle  $ASC$  we have  $CD/SD = CF/FA = 3/2$ . But  $DC = BD/3 = (BS + SD)/3$ . Therefore  $BS/SD = 7/2$ .  $\square$

5. Let  $a_1, b_1, c_1$  be natural numbers. We define

$$a_2 = \gcd(b_1, c_1), \quad b_2 = \gcd(c_1, a_1), \quad c_2 = \gcd(a_1, b_1),$$

and

$$a_3 = \text{lcm}(b_2, c_2), \quad b_3 = \text{lcm}(c_2, a_2), \quad c_3 = \text{lcm}(a_2, b_2).$$

Show that  $\gcd(b_3, c_3) = a_2$ .

**Solution.** For a prime  $p$  and a natural number  $n$  we shall denote by  $v_p(n)$  the power of  $p$  dividing  $n$ . Then it is enough to show that  $v_p(a_2) = v_p(\gcd(b_3, c_3))$  for all primes  $p$ . Let  $p$  be a prime and let  $\alpha = v_p(a_1), \beta = v_p(b_1)$  and  $\gamma = v_p(c_1)$ . Because of symmetry, we may assume that  $\alpha \leq \beta \leq \gamma$ . Therefore,  $v_p(a_2) = \min\{\beta, \gamma\} = \beta$  and similarly  $v_p(b_2) = v_p(c_2) = \alpha$ . Therefore  $v_p(b_3) = \max\{\alpha, \beta\} = \beta$  and similarly  $v_p(c_3) = \max\{\alpha, \beta\} = \beta$ . Therefore  $v_p(\gcd(b_3, c_3)) = v_p(a_2) = \beta$ . This completes the solution.  $\square$

6. Let  $a, b$  be real numbers and, let  $P(x) = x^3 + ax^2 + b$  and  $Q(x) = x^3 + bx + a$ . Suppose that the roots of the equation  $P(x) = 0$  are the reciprocals of the roots of the equation  $Q(x) = 0$ . Find the greatest common divisor of  $P(2013! + 1)$  and  $Q(2013! + 1)$ .

**Solution.** Note that  $P(0) \neq 0$ . Let  $R(x) = x^3 P(1/x) = bx^3 + ax + 1$ . Then the equations  $Q(x) = 0$  and  $R(x) = 0$  have the same roots. This implies that  $R(x) = bQ(x)$  and equating the coefficients we get  $a = b^2$  and  $ab = 1$ . This implies that  $b^3 = 1$ , so  $a = b = 1$ . Thus  $P(x) = x^3 + x^2 + 1$  and  $Q(x) = x^3 + x + 1$ . For any integer  $n$  we have

$$(P(n), Q(n)) = (P(n), P(n) - Q(n)) = (n^3 + n^2 + 1, n^2 - n) = (n^3 + n^2 + 1, n - 1) = (3, n - 1).$$

Thus  $(P(n), Q(n)) = 3$  if  $n - 1$  is divisible by 3. In particular, since 3 divides  $2013!$  it follows that  $(P(2013! + 1), Q(2013! + 1)) = 3$ .

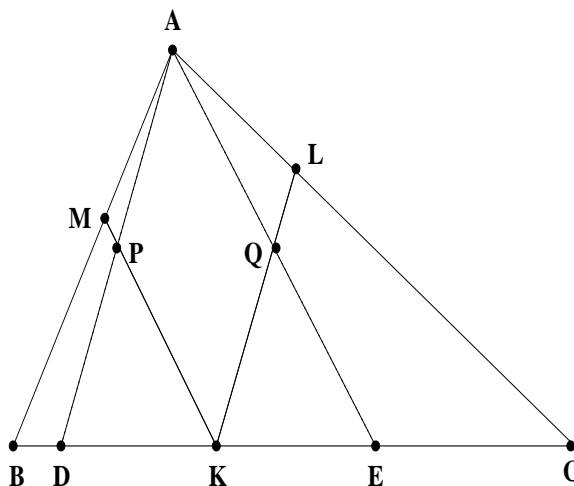
$\square$

# INMO-2000

## Problems and Solutions

1. The in-circle of triangle  $ABC$  touches the sides  $BC$ ,  $CA$  and  $AB$  in  $K$ ,  $L$  and  $M$  respectively. The line through  $A$  and parallel to  $LK$  meets  $MK$  in  $P$  and the line through  $A$  and parallel to  $MK$  meets  $LK$  in  $Q$ . Show that the line  $PQ$  bisects the sides  $AB$  and  $AC$  of triangle  $ABC$ .

**Solution.** : Let  $AP, AQ$  produced meet  $BC$  in  $D, E$  respectively.



Since  $MK$  is parallel to  $AE$ , we have  $\angle AEK = \angle MKB$ . Since  $BK = BM$ , both being tangents to the circle from  $B$ ,  $\angle MKB = \angle BMK$ . This with the fact that  $MK$  is parallel to  $AE$  gives us  $\angle AEK = \angle MAE$ . This shows that  $MAEK$  is an isosceles trapezoid. We conclude that  $MA = KE$ . Similarly, we can prove that  $AL = DK$ . But  $AM = AL$ . We get that  $DK = KE$ . Since  $KP$  is parallel to  $AE$ , we get  $DP = PA$  and similarly  $EQ = QA$ . This implies that  $PQ$  is parallel to  $DE$  and hence bisects  $AB, AC$  when produced.

[The same argument holds even if one or both of  $P$  and  $Q$  lie outside triangle  $ABC$ .]

2. Solve for integers  $x, y, z$ :

$$x + y = 1 - z, \quad x^3 + y^3 = 1 - z^2.$$

**Sol.** : Eliminating  $z$  from the given set of equations, we get

$$x^3 + y^3 + \{1 - (x + y)\}^2 = 1.$$

This factors to

$$(x+y)(x^2 - xy + y^2 + x + y - 2) = 0.$$

**Case 1.** Suppose  $x + y = 0$ . Then  $z = 1$  and  $(x, y, z) = (m, -m, 1)$ , where  $m$  is an integer give one family of solutions.

**Case 2.** Suppose  $x + y \neq 0$ . Then we must have

$$x^2 - xy + y^2 + x + y - 2 = 0.$$

This can be written in the form

$$(2x - y + 1)^2 + 3(y + 1)^2 = 12.$$

Here there are two possibilities:

$$2x - y + 1 = 0, y + 1 = \pm 2; \quad 2x - y + 1 = \pm 3, y + 1 = \pm 1.$$

Analysing all these cases we get

$$(x, y, z) = (0, 1, 0), (-2, -3, 6), (1, 0, 0), (0, -2, 3), (-2, 0, 3), (-3, -2, 6).$$

3. If  $a, b, c, x$  are real numbers such that  $abc \neq 0$  and

$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c},$$

then prove that either  $a + b + c = 0$  or  $a = b = c$ .

**Sol. :** Suppose  $a + b + c \neq 0$  and let the common value be  $\lambda$ . Then

$$\lambda = \frac{xb + (1-x)c + xc + (1-x)a + xa + (1-x)b}{a + b + c} = 1.$$

We get two equations:

$$-a + xb + (1-x)c = 0, \quad (1-x)a - b + xc = 0.$$

(The other equation is a linear combination of these two.) Using these two equations, we get the relations

$$\frac{a}{1-x+x^2} = \frac{b}{x^2-x+1} = \frac{c}{(1-x)^2+x}.$$

Since  $1-x+x^2 \neq 0$ , we get  $a = b = c$ .

4. In a convex quadrilateral  $PQRS$ ,  $PQ = RS$ ,  $(\sqrt{3}+1)QR = SP$  and  $\angle RSP - \angle SPQ = 30^\circ$ . Prove that

$$\angle PQR - \angle QRS = 90^\circ.$$

**Sol. :** Let  $[\text{Fig}]$  denote the area of Fig. We have

$$[PQRS] = [PQR] + [RSP] = [QRS] + [SPQ].$$

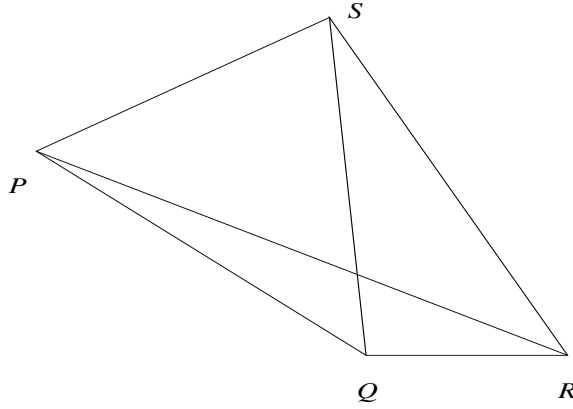
Let us write  $PQ = p, QR = q, RS = r, SP = s$ . The above relations reduce to

$$pq \sin \angle PQR + rs \sin \angle RSP = qr \sin \angle QRS + sp \sin \angle SPQ.$$

Using  $p = r$  and  $(\sqrt{3} + 1)q = s$  and dividing by  $pq$ , we get

$$\sin \angle PQR + (\sqrt{3} + 1) \sin \angle RSP = \sin \angle QRS + (\sqrt{3} + 1) \sin \angle SPQ.$$

Therefore,  $\sin \angle PQR - \sin \angle QRS = (\sqrt{3} + 1)(\sin \angle SPQ - \sin \angle RSP)$ .



**Fig. 2.**

This can be written in the form

$$\begin{aligned} 2 \sin \frac{\angle PQR - \angle QRS}{2} \cos \frac{\angle PQR + \angle QRS}{2} \\ = (\sqrt{3} + 1) 2 \sin \frac{\angle SPQ - \angle RSP}{2} \cos \frac{\angle SPQ + \angle RSP}{2}. \end{aligned}$$

Using the relations

$$\cos \frac{\angle PQR + \angle QRS}{2} = -\cos \frac{\angle SPQ + \angle RSP}{2}$$

and

$$\sin \frac{\angle SPQ - \angle RSP}{2} = -\sin 15^\circ = -\frac{(\sqrt{3} - 1)}{2\sqrt{2}},$$

we obtain

$$\sin \frac{\angle PQR - \angle QRS}{2} = (\sqrt{3} + 1) \left[ -\frac{(\sqrt{3} - 1)}{2\sqrt{2}} \right] = \frac{1}{\sqrt{2}}.$$

This shows that

$$\frac{\angle PQR - \angle QRS}{2} = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}.$$

Using the convexity of  $PQRS$ , we can rule out the latter alternative. We obtain

$$\angle PQR - \angle QRS = \frac{\pi}{2}.$$

5. Let  $a, b, c$  be three real numbers such that  $1 \geq a \geq b \geq c \geq 0$ . Prove that if  $\lambda$  is a root of the cubic equation  $x^3 + ax^2 + bx + c = 0$  (real or complex), then  $|\lambda| \leq 1$ .

**Sol. :** Since  $\lambda$  is a root of the equation  $x^3 + ax^2 + bx + c = 0$ , we have

$$\lambda^3 = -a\lambda^2 - b\lambda - c.$$

This implies that

$$\begin{aligned} \lambda^4 &= -a\lambda^3 - b\lambda^2 - c\lambda \\ &= (1-a)\lambda^3 + (a-b)\lambda^2 + (b-c)\lambda + c \end{aligned}$$

where we have used again

$$-\lambda^3 - a\lambda^2 - b\lambda - c = 0.$$

Suppose  $|\lambda| \geq 1$ . Then we obtain

$$\begin{aligned} |\lambda|^4 &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^2 + (b-c)|\lambda| + c \\ &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^3 + (b-c)|\lambda|^3 + c|\lambda|^3 \\ &\leq |\lambda|^3. \end{aligned}$$

This shows that  $|\lambda| \leq 1$ . Hence the only possibility in this case is  $|\lambda| = 1$ . We conclude that  $|\lambda| \leq 1$  is always true.

6. For any natural number  $n$ , ( $n \geq 3$ ), let  $f(n)$  denote the number of non-congruent integer-sided triangles with perimeter  $n$  (e.g.,  $f(3) = 1, f(4) = 0, f(7) = 2$ ). Show that

$$(a) \quad f(1999) > f(1996);$$

$$(b) \quad f(2000) = f(1997).$$

**Sol. :**

(a) Let  $a, b, c$  be the sides of a triangle with  $a + b + c = 1996$ , and each being a positive integer. Then  $a + 1, b + 1, c + 1$  are also sides of a triangle with perimeter 1999 because

$$a < b + c \implies a + 1 < (b + 1) + (c + 1),$$

and so on. Moreover  $(999, 999, 1)$  form the sides of a triangle with perimeter 1999, which is not obtainable in the form  $(a+1, b+1, c+1)$  where  $a, b, c$  are the integers and the sides of a triangle with  $a + b + c = 1996$ . We conclude that  $f(1999) > f(1996)$ .

(b) As in the case (a) we conclude that  $f(2000) \geq f(1997)$ . On the other hand, if  $x, y, z$  are the integer sides of a triangle with  $x + y + z = 2000$ , and say  $x \geq y \geq z \geq 1$ , then we cannot have  $z = 1$ ; for otherwise we would get  $x + y = 1999$  forcing  $x, y$  to have opposite parity so that  $x - y \geq 1 = z$  violating triangle inequality for  $x, y, z$ . Hence  $x \geq y \geq z > 1$ . This implies that  $x - 1 \geq y - 1 \geq z - 1 > 0$ . We already have  $x < y + z$ . If  $x \geq y + z - 1$ , then we see that  $y + z - 1 \leq x < y + z$ , showing that  $y + z - 1 = x$ . Hence we obtain  $2000 = x + y + z = 2x + 1$  which is impossible. We conclude that  $x < y + z - 1$ . This shows that  $x - 1 < (y - 1) + (z - 1)$  and hence  $x - 1, y - 1, z - 1$  are the sides of a triangle with perimeter 1997. This gives  $f(2000) \leq f(1997)$ . Thus we obtain the desired result.

---

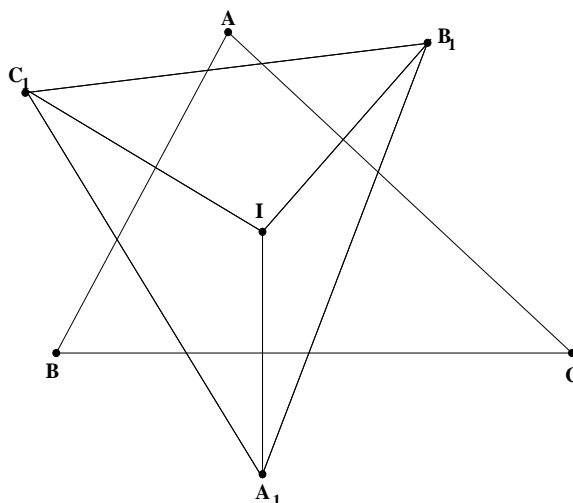
# INMO-2001

## Problems and Solutions

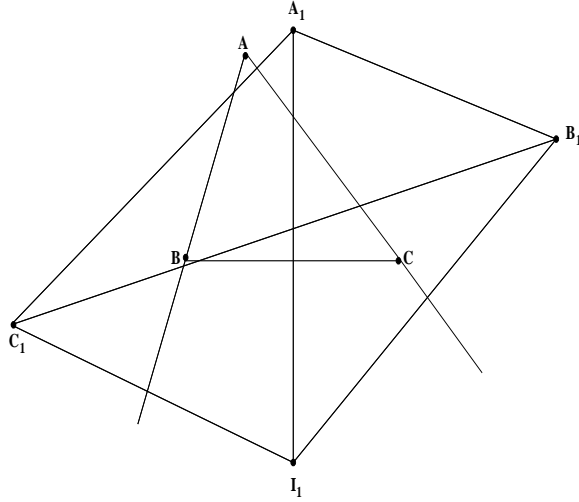
1. Let  $ABC$  be a triangle in which *no* angle is  $90^\circ$ . For any point  $P$  in the plane of the triangle, let  $A_1, B_1, C_1$  denote the reflections of  $P$  in the sides  $BC, CA, AB$  respectively. Prove the following statements:
  - (a) If  $P$  is the incentre or an excentre of  $ABC$ , then  $P$  is the circumcentre of  $A_1B_1C_1$ ;
  - (b) If  $P$  is the circumcentre of  $ABC$ , then  $P$  is the orthocentre of  $A_1B_1C_1$ ;
  - (c) If  $P$  is the orthocentre of  $ABC$ , then  $P$  is either the incentre or an excentre of  $A_1B_1C_1$ .

**Solution:**

(a)



If  $P = I$  is the incentre of triangle  $ABC$ , and  $r$  its inradius, then it is clear that  $A_1I = B_1I = C_1I = 2r$ . It follows that  $I$  is the circumcentre of  $A_1B_1C_1$ . On the otherhand if  $P = I_1$  is the excentre of  $ABC$  opposite  $A$  and  $r_1$  the corresponding exradius, then again we see that  $A_1I_1 = B_1I_1 = C_1I_1 = 2r_1$ . Thus  $I_1$  is the circumcentre of  $A_1B_1C_1$ .



(b)

Let  $P = O$  be the circumcentre of  $ABC$ . By definition, it follows that  $OA_1$  bisects  $BC$  and is bisected by  $BC$  and so on. Let  $D, E, F$  be the mid-points of  $BC, CA, AB$  respectively. Then  $FE$  is parallel to  $BC$ . But  $E, F$  are also mid-points of  $OB_1, OC_1$  and hence  $FE$  is parallel to  $B_1C_1$  as well. We conclude that  $BC$  is parallel to  $B_1C_1$ . Since  $OA_1$  is perpendicular to  $BC$ , it follows that  $OA_1$  is perpendicular to  $B_1C_1$ . Similarly  $OB_1$  is perpendicular to  $C_1A_1$  and  $OC_1$  is perpendicular to  $A_1B_1$ . These imply that  $O$  is the orthocentre of  $A_1B_1C_1$ . (This applies whether  $O$  is inside or outside  $ABC$ .)

(c)

let  $P = H$ , the orthocentre of  $ABC$ . We consider two possibilities;  $H$  falls inside  $ABC$  and  $H$  falls outside  $ABC$ .

Suppose  $H$  is inside  $ABC$ ; this happens if  $ABC$  is an acute triangle. It is known that  $A_1, B_1, C_1$  lie on the circumcircle of  $ABC$ . Thus  $\angle C_1A_1A = \angle C_1CA = 90^\circ - A$ . Similarly  $\angle B_1A_1A = \angle B_1BA = 90^\circ - A$ . These show that  $\angle C_1A_1A = \angle B_1A_1A$ . Thus  $A_1A$  is an internal bisector of  $\angle C_1A_1B_1$ . Similarly we can show that  $B_1$  bisects  $\angle A_1B_1C_1$  and  $C_1C$  bisects  $\angle B_1C_1A_1$ . Since  $A_1A, B_1B, C_1C$  concur at  $H$ , we conclude that  $H$  is the incentre of  $A_1B_1C_1$ .

**OR** If  $D, E, F$  are the feet of perpendiculars of  $A, B, C$  to the sides  $BC, CA, AB$  respectively, then we see that  $EF, FD, DE$  are respectively parallel to  $B_1C_1, C_1A_1, A_1B_1$ . This implies that  $\angle C_1A_1H = \angle FDH = \angle ABE = 90^\circ - A$ , as  $BDHF$  is a cyclic quadrilateral. Similarly, we can show that  $\angle B_1A_1H = 90^\circ - A$ . It follows that  $A_1H$  is the internal bisector of  $\angle C_1A_1B_1$ . We can proceed as in the earlier case.

If  $H$  is outside  $ABC$ , the same proofs go through again, except that two of  $A_1H, B_1H, C_1H$  are external angle bisectors and one of these is an internal angle bisector. Thus  $H$  becomes an excentre of triangle  $A_1B_1C_1$ .

2. Show that the equation

$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$$

has infinitely many solutions in integers  $x, y, z$ .

**Solution:** We seek solutions  $(x, y, z)$  which are in arithmetic progression. Let us put  $y - x = z - y = d > 0$  so that the equation reduces to the form

$$3y^2 + 2d^2 = 2d^3.$$

Thus we get  $3y^2 = 2(d - 1)d^2$ . We conclude that  $2(d - 1)$  is 3 times a square. This is satisfied if  $d - 1 = 6n^2$  for some  $n$ . Thus  $d = 6n^2 + 1$  and  $3y^2 = d^2 \cdot 2(6n^2)$  giving us  $y^2 = 4d^2n^2$ . Thus we can take  $y = 2dn = 2n(6n^2 + 1)$ . From this we obtain  $x = y - d = (2n - 1)(6n^2 + 1)$ ,  $z = y + d = (2n + 1)(6n^2 + 1)$ . It is easily verified that

$$(x, y, z) = ((2n - 1)(6n^2 + 1), 2n(6n^2 + 1), (2n + 1)(6n^2 + 1)),$$

is indeed a solution for a fixed  $n$  and this gives an infinite set of solutions as  $n$  varies over natural numbers.

3. If  $a, b, c$  are positive real numbers such that  $abc = 1$ , prove that

$$a^{b+c} b^{c+a} c^{a+b} \leq 1.$$

**Solution:** Note that the inequality is symmetric in  $a, b, c$  so that we may assume that  $a \geq b \geq c$ . Since  $abc = 1$ , it follows that  $a \geq 1$  and  $c \leq 1$ . Using  $b = 1/ac$ , we get

$$a^{b+c} b^{c+a} c^{a+b} = \frac{a^{b+c} c^{a+b}}{a^{c+a} c^{c+a}} = \frac{c^{b-c}}{a^{a-b}} \leq 1,$$

because  $c \leq 1$ ,  $b \geq c$ ,  $a \geq 1$  and  $a \geq b$ .

4. Given any nine integers show that it is possible to choose, from among them, four integers  $a, b, c, d$  such that  $a + b - c - d$  is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

**Solution:**

Suppose there are four numbers  $a, b, c, d$  among the given nine numbers which leave the same remainder modulo 20. Then  $a + b \equiv c + d \pmod{20}$  and we are done.

If not, there are two possibilities:

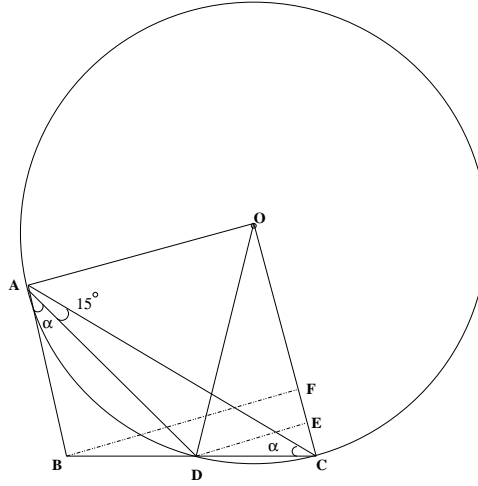
- (1) We may have two disjoint pairs  $\{a, c\}$  and  $\{b, d\}$  obtained from the given nine numbers such that  $a \equiv c \pmod{20}$  and  $b \equiv d \pmod{20}$ . In this case we get  $a + b \equiv c + d \pmod{20}$ .

(2) Or else there are at most three numbers having the same remainder modulo 20 and the remaining six numbers leave distinct remainders which are also different from the first remainder (i.e., the remainder of the three numbers). Thus there are at least 7 distinct remainders modulo 20 that can be obtained from the given set of nine numbers. These 7 remainders give rise to  $\binom{7}{2} = 21$  pairs of numbers. By pigeonhole principle, there must be two pairs  $(r_1, r_2), (r_3, r_4)$  such that  $r_1 + r_2 \equiv r_3 + r_4 \pmod{20}$ . Going back we get four numbers  $a, b, c, d$  such that  $a + b \equiv c + d \pmod{20}$ .

If we take the numbers 0, 0, 0, 1, 2, 4, 7, 12, we check that the result is not true for these eight numbers.

5. Let  $ABC$  be a triangle and  $D$  be the mid-point of side  $BC$ . Suppose  $\angle DAB = \angle BCA$  and  $\angle DAC = 15^\circ$ . Show that  $\angle ADC$  is obtuse. Further, if  $O$  is the circumcentre of  $ADC$ , prove that triangle  $AOD$  is equilateral.

**Solution:**



Let  $\alpha$  denote the equal angles  $\angle BAD = \angle DCA$ . Using sine rule in triangles  $DAB$  and  $DAC$ , we get

$$\frac{AD}{\sin B} = \frac{BD}{\sin \alpha}, \quad \frac{CD}{\sin 15^\circ} = \frac{AD}{\sin \alpha}.$$

Eliminating  $\alpha$  (using  $BD = DC$  and  $2\alpha + B + 15^\circ = \pi$ ), we obtain  $1 + \cos(B + 15^\circ) = 2 \sin B \sin 15^\circ$ . But we know that  $2 \sin B \sin 15^\circ = \cos(B - 15^\circ) - \cos(B + 15^\circ)$ . Putting  $\beta = B - 15^\circ$ , we get a relation  $1 + 2 \cos(\beta + 30) = \cos \beta$ . We write this in the form

$$(1 - \sqrt{3}) \cos \beta + \sin \beta = 1.$$

Since  $\sin \beta \leq 1$ , it follows that  $(1 - \sqrt{3}) \cos \beta \geq 0$ . We conclude that  $\cos \beta \leq 0$  and hence that  $\beta$  is obtuse. So is angle  $B$  and hence  $\angle ADC$ .

We have the relation  $(1 - \sqrt{3}) \cos \beta + \sin \beta = 1$ . If we set  $x = \tan(\beta/2)$ , then we get, using  $\cos \beta = (1 - x^2)/(1 + x^2)$ ,  $\sin \beta = 2x/(1 + x^2)$ ,

$$(\sqrt{3} - 2)x^2 + 2x - \sqrt{3} = 0.$$

Solving for  $x$ , we obtain  $x = 1$  or  $x = \sqrt{3}(2 + \sqrt{3})$ . If  $x = \sqrt{3}(2 + \sqrt{3})$ , then  $\tan(\beta/2) > 2 + \sqrt{3} = \tan 75^\circ$  giving us  $\beta > 150^\circ$ . This forces that  $B > 165^\circ$  and hence  $B + A > 165^\circ + 15^\circ = 180^\circ$ , a contradiction. thus  $x = 1$  giving us  $\beta = \pi/2$ . This gives  $B = 105^\circ$  and hence  $\alpha = 30^\circ$ . Thus  $\angle DAO = 60^\circ$ . Since  $OA = OD$ , the result follows.

**OR**

Let  $m_a$  denote the median  $AD$ . Then we can compute

$$\cos \alpha = \frac{c^2 + m_a^2 - (a^2/4)}{2cm_a}, \quad \sin \alpha = \frac{2\Delta}{cm_a},$$

where  $\Delta$  denotes the area of triangle  $ABC$ . These two expressions give

$$\cot \alpha = \frac{c^2 + m_a^2 - (a^2/4)}{4\Delta}.$$

Similarly, we obtain

$$\cot \angle CAD = \frac{b^2 + m_a^2 - (a^2/4)}{4\Delta}.$$

Thus we get

$$\cot \alpha - \cot 15^\circ = \frac{c^2 - a^2}{4\Delta}.$$

Similarly we can also obtain

$$\cot B - \cot \alpha = \frac{c^2 - a^2}{4\Delta},$$

giving us the relation

$$\cot B = 2 \cot \alpha - \cot 15^\circ.$$

If  $B$  is acute then  $2 \cot \alpha > \cot 15^\circ = 2 + \sqrt{3} > 2\sqrt{3}$ . It follows that  $\cot \alpha > \sqrt{3}$ . This implies that  $\alpha < 30^\circ$  and hence

$$B = 180^\circ - 2\alpha - 15^\circ > 105^\circ.$$

This contradiction forces that angle  $B$  is obtuse and consequently  $\angle ADC$  is obtuse.

Since  $\angle BAD = \alpha = \angle ACD$ , the line  $AB$  is tangent to the circumcircle  $\Gamma$  of  $ADC$  at  $A$ . Hence  $OA$  is perpendicular to  $AB$ . Draw  $DE$  and  $BF$  perpendicular to  $AC$ , and join  $OD$ . Since  $\angle DAC = 15^\circ$ , we see that  $\angle DOC = 30^\circ$  and hence  $DE = OD/2$ . But  $DE$  is parallel to  $BF$  and  $BD = DC$  shows that  $BF = 2DE$ . We conclude that

$BF = DO$ . But  $DO = AO$ , both being radii of  $\Gamma$ . Thus  $BF = AO$ . Using right triangles  $BFO$  and  $BAO$ , we infer that  $AB = OF$ . We conclude that  $ABFO$  is a rectangle. In particular  $\angle AOF = 90^\circ$ . It follows that

$$\angle AOD = 90^\circ - \angle DOC = 90^\circ - 30^\circ = 60^\circ.$$

Since  $OA = OD$ , we conclude that  $AOD$  is equilateral.

**OR**

Note that triangles  $ABD$  and  $CBA$  are similar. Thus we have the ratios

$$\frac{AB}{BD} = \frac{CB}{BA}.$$

This reduces to  $a^2 = 2c^2$  giving us  $a = \sqrt{2}c$ . This is equivalent to  $\sin^2(\alpha + 15^\circ) = 2\sin^2\alpha$ . We write this in the form

$$\cos 15^\circ + \cot \alpha \sin 15^\circ = \sqrt{2}.$$

Solving for  $\cot \alpha$ , we get  $\cot \alpha = \sqrt{3}$ . We conclude that  $\alpha = 30^\circ$ , and the result follows.

6. Let  $\mathbf{R}$  denote the set of all real numbers. Find all functions  $f : \mathbf{R} \rightarrow \mathbf{R}$  satisfying the condition

$$f(x+y) = f(x)f(y)f(xy)$$

for all  $x, y$  in  $\mathbf{R}$ .

**Solution:** Putting  $x = 0, y = 0$ , we get  $f(0) = f(0)^3$  so that  $f(0) = 0, 1$  or  $-1$ . If  $f(0) = 0$ , then taking  $y = 0$  in the given equation, we obtain  $f(x) = f(x)f(0)^2 = 0$  for all  $x$ .

Suppose  $f(0) = 1$ . Taking  $y = -x$ , we obtain

$$1 = f(0) = f(x-x) = f(x)f(-x)f(-x^2).$$

This shows that  $f(x) \neq 0$  for any  $x \in \mathbf{R}$ . Taking  $x = 1, y = x - 1$ , we obtain

$$f(x) = f(1)f(x-1)^2 = f(1)[f(x)f(-x)f(-x)]^2.$$

Using  $f(x) \neq 0$ , we conclude that  $1 = kf(x)(f(-x))^2$ , where  $k = f(1)(f(-1))^2$ . Changing  $x$  to  $-x$  here, we also infer that  $1 = kf(-x)(f(x))^2$ . Comparing these expressions we see that  $f(-x) = f(x)$ . It follows that  $1 = kf(x)^3$ . Thus  $f(x)$  is constant for all  $x$ . Since  $f(0) = 1$ , we conclude that  $f(x) = 1$  for all real  $x$ .

If  $f(0) = -1$ , a similar analysis shows that  $f(x) = -1$  for all  $x \in \mathbf{R}$ . We can verify that each of these functions satisfies the given functional equation. Thus there are three solutions, all of them being constant functions.

## Solution to INMO-2002 Problems

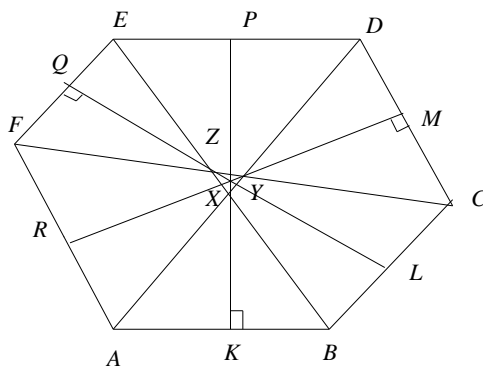
1. For a convex hexagon  $ABCDEF$  in which each pair of opposite sides is unequal, consider the following six statements:

$$\begin{array}{ll} (a_1) \ AB \text{ is parallel to } DE; & (a_2) \ AE = BD; \\ (b_1) \ BC \text{ is parallel to } EF; & (b_2) \ BF = CE; \\ (c_1) \ CD \text{ is parallel to } FA; & (c_2) \ CA = DF. \end{array}$$

- (a) Show that if all the six statements are true, then the hexagon is cyclic (i.e., it can be inscribed in a circle).  
 (b) Prove that, in fact, any five of these six statements also imply that the hexagon is cyclic.

### Solution:

(a) Suppose all the six statements are true. Then  $ABDE$ ,  $BCEF$ ,  $C DFA$  are isosceles trapeziums; if  $K, L, M, P, Q, R$  are the mid-points of  $AB, BC, CD, DE, EF, FA$  respectively, then we see that  $KP \perp AB, ED$ ;  $LQ \perp BC, EF$  and  $MR \perp CD, FA$ .



If  $AD, BE, CF$  themselves concur at a point  $O$ , then  $OA = OB = OC = OD = OE = OF$ . ( $O$  is on the perpendicular bisector of each of the sides.) Hence  $A, B, C, D, E, F$  are concyclic and lie on a circle with centre  $O$ . Otherwise these lines  $AD, BE, CF$  form a triangle, say  $XYZ$ . (See Fig.) Then  $KX, MY, QZ$ , when extended, become the internal angle bisectors of the triangle  $XYZ$  and hence concur at the incentre  $O'$  of  $XYZ$ . As earlier  $O'$  lies on the perpendicular bisector of each of the sides. Hence  $O'A = O'B = O'C = O'D = O'E = O'F$ , giving the concyclicity of  $A, B, C, D, E, F$ .

(b) Suppose  $(a_1)$ ,  $(a_2)$ ,  $(b_1)$ ,  $(b_2)$  are true. Then we see that  $AD = BE = CF$ . Assume that  $(c_1)$  is true. Then  $CD$  is parallel to  $AF$ . It follows that triangles  $YCD$  and  $YFA$  are similar. This gives

$$\frac{FY}{AY} = \frac{YC}{YD} = \frac{FY + YC}{AY + YD} = \frac{FC}{AD} = 1.$$

We obtain  $FY = AY$  and  $YC = YD$ . This forces that triangles  $CYA$  and  $DYF$  are congruent. In particular  $AC = DF$  so that  $(c_2)$  is true. The conclusion follows from (a). Now assume that  $(c_2)$  is true; i.e.,  $AC = FD$ . We have seen that  $AD = BE = CF$ . It follows that triangles  $FDC$  and  $ACD$  are congruent. In particular  $\angle ADC = \angle FCD$ . Similarly, we can show that  $\angle CFA = \angle DAF$ . We conclude that  $CD$  is parallel to  $AF$  giving  $(c_1)$ .

2. Determine the least positive value taken by the expression  $a^3 + b^3 + c^3 - 3abc$  as  $a, b, c$  vary over all positive integers. Find also all triples  $(a, b, c)$  for which this least value is attained.

**Solution:** We observe that

$$Q = a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c) \left( (a - b)^2 + (b - c)^2 + (c - a)^2 \right).$$

Since we are looking for the least positive value taken by  $Q$ , it follows that  $a, b, c$  are not all equal. Thus  $a + b + c \geq 1 + 1 + 2 = 4$  and  $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 1 + 1 + 0 = 2$ . Thus we see that  $Q \geq 4$ . Taking  $a = 1$ ,  $b = 1$  and  $c = 2$ , we get  $Q = 4$ . Therefore the least value of  $Q$  is 4 and this is achieved only by  $a + b + c = 4$  and  $(a - b)^2 + (b - c)^2 + (c - a)^2 = 2$ . The triples for which  $Q = 4$  are therefore given by

$$(a, b, c) = (1, 1, 2), (1, 2, 1), (2, 1, 1).$$

3. Let  $x, y$  be positive reals such that  $x + y = 2$ . Prove that

$$x^3 y^3 (x^3 + y^3) \leq 2.$$

**Solution:** We have from the AM-GM inequality, that

$$xy \leq \left( \frac{x + y}{2} \right)^2 = 1.$$

Thus we obtain  $0 < xy \leq 1$ . We write

$$\begin{aligned} x^3 y^3 (x^3 + y^3) &= (xy)^3 (x + y) (x^2 - xy + y^2) \\ &= 2(xy)^3 \left( (x + y)^2 - 3xy \right) \\ &= 2(xy)^3 (4 - 3xy). \end{aligned}$$

Thus we need to prove that

$$(xy)^3(4 - 3xy) \leq 1.$$

Putting  $z = xy$ , this inequality reduces to

$$z^3(4 - 3z) \leq 1,$$

for  $0 < z \leq 1$ . We can prove this in different ways. We can put the inequality in the form

$$3z^4 - 4z^3 + 1 \geq 0.$$

Here the expression in the **LHS** factors to  $(z - 1)^2(3z^2 + 2z + 1)$  and  $(3z^2 + 2z + 1)$  is positive since its discriminant  $D = -8 < 0$ . Or applying the AM-GM inequality to the positive reals  $4 - 3z, z, z, z$ , we obtain

$$z^3(4 - 3z) \leq \left( \frac{4 - 3z + 3z}{4} \right)^4 \leq 1.$$

4. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?

**Solution:** Any set of 100 lines in the plane can be partitioned into a finite number of disjoint sets, say  $A_1, A_2, A_3, \dots, A_k$ , such that

- (i) Any two lines in each  $A_j$  are parallel to each other, for  $1 \leq j \leq k$  (provided, of course,  $|A_j| \geq 2$ );
- (ii) for  $j \neq l$ , the lines in  $A_j$  and  $A_l$  are not parallel.

If  $|A_j| = m_j$ ,  $1 \leq j \leq k$ , then the total number of points of intersection is given by  $\sum_{1 \leq j < l \leq k} m_j m_l$ , as no three lines are concurrent. Thus we have to find positive integers  $m_1, m_2, \dots, m_k$  such that

$$\sum_{j=1}^k m_j = 100, \quad \sum_{j=1}^k m_j m_l = 2002,$$

for an affirmative answer to the given question.

We observe that

$$\begin{aligned} \sum_{j=1}^k m_j^2 &= \left( \sum_{j=1}^k m_j \right)^2 - 2 \left( \sum_{j=1}^k m_j m_l \right) \\ &= 100^2 - 2(2002) = 5996. \end{aligned}$$

Thus we have to choose  $m_1, m_2, \dots, m_k$  such that

$$\sum_{j=1}^k m_j = 100, \quad \sum_{j=1}^k m_j^2 = 5996.$$

We observe that  $\lceil \sqrt{5996} \rceil = 77$ . So we may take  $m_1 = 77$ , so that

$$\sum_{j=2}^k m_j = 23, \quad \sum_{j=2}^k m_j^2 = 67.$$

Now we may choose  $m_2 = 5$ ,  $m_3 = m_4 = 4$ ,  $m_5 = m_6 = \dots = m_{14} = 1$ . Finally, we can take

$$k = 14, \quad (m_1, m_2, \dots, m_{14}) = (77, 5, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

proving the existence of 100 lines with exactly 2002 points of intersection.

5. Do there exist three distinct positive real numbers  $a, b, c$  such that the numbers  $a, b, c, b + c - a, c + a - b, a + b - c$  and  $a + b + c$  form a 7-term arithmetic progression in some order?

**Solution:** We show that the answer is **NO**. Suppose, if possible, let  $a, b, c$  be three distinct positive real numbers such that  $a, b, c, b + c - a, c + a - b, a + b - c$  and  $a + b + c$  form a 7-term arithmetic progression in some order. We may assume that  $a < b < c$ . Then there are only two cases we need to check: (I)  $a + b - c < a < c + a - b < b < c < b + c - a < a + b + c$  and (II)  $a + b - c < a < b < c + a - b < c < b + c - a < a + b + c$ .

**Case I.** Suppose the chain of inequalities  $a + b - c < a < c + a - b < b < c < b + c - a < a + b + c$  holds good. let  $d$  be the common difference. Thus we see that

$$c = a + b + c - 2d, \quad b = a + b + c - 3d, \quad a = a + b + c - 5d.$$

Adding these, we see that  $a + b + c = 5d$ . But then  $a = 0$  contradicting the positivity of  $a$ .

**Case II.** Suppose the inequalities  $a + b - c < a < b < c + a - b < c < b + c - a < a + b + c$  are true. Again we see that

$$c = a + b + c - 2d, \quad b = a + b + c - 4d, \quad a = a + b + c - 5d.$$

We thus obtain  $a + b + c = (11/2)d$ . This gives

$$a = \frac{1}{2}d, \quad b = \frac{3}{2}d, \quad c = \frac{7}{2}d.$$

Note that  $a + b - c = a + b + c - 6d = -(1/2)d$ . However we also get  $a + b - c = [(1/2) + (3/2) - (7/2)]d = -(3/2)d$ . It follows that  $3e = e$  giving  $d = 0$ . But this is impossible.

Thus there are no three distinct positive real numbers  $a, b, c$  such that  $a, b, c, b + c - a, c + a - b, a + b - c$  and  $a + b + c$  form a 7-term arithmetic progression in some order.

6. Suppose the  $n^2$  numbers  $1, 2, 3, \dots, n^2$  are arranged to form an  $n$  by  $n$  array consisting of  $n$  rows and  $n$  columns such that the numbers in each row (from left to right) and each column (from top to bottom) are in increasing order. Denote by  $a_{jk}$  the number in  $j$ -th row and  $k$ -th column. Suppose  $b_j$  is the maximum possible number of entries that can occur as  $a_{jj}$ ,  $1 \leq j \leq n$ . Prove that

$$b_1 + b_2 + b_3 + \dots + b_n \leq \frac{n}{3}(n^2 - 3n + 5).$$

(Example: In the case  $n = 3$ , the only numbers which can occur as  $a_{22}$  are 4, 5 or 6 so that  $b_2 = 3$ .)

**Solution:** Since  $a_{jj}$  has to exceed all the numbers in the top left  $j \times j$  submatrix (excluding itself), and since there are  $j^2 - 1$  entries, we must have  $a_{jj} \geq j^2$ . Similarly,  $a_{jj}$  must not exceed eac of the numbers in the bottom right  $(n - j + 1) \times (n - j + 1)$  submatrix (other than itself) and there are  $(n - j + 1)^2 - 1$  such entries giving  $a_{jj} \leq n^2 - (n - j + 1)^2 + 1$ . Thus we see that

$$a_{jj} \in \{j^2, j^2 + 1, j^2 + 2, \dots, n^2 - (n - j + 1)^2 + 1\}.$$

The number of elements in this set is  $n^2 - (n - j + 1)^2 - j^2 + 2$ . This implies that

$$b_j \leq n^2 - (n - j + 1)^2 - j^2 + 2 = (2n + 2)j - 2j^2 - (2n - 1).$$

It follows that

$$\begin{aligned} \sum_{j=1}^n b_j &\leq (2n + 2) \sum_{j=1}^n j - 2 \sum_{j=1}^n j^2 - n(2n - 1) \\ &= (2n + 2) \left( \frac{n(n + 1)}{2} \right) - 2 \left( \frac{n(n + 1)(2n + 1)}{6} \right) - n(2n - 1) \\ &= \frac{n}{3}(n^2 - 3n + 5), \end{aligned}$$

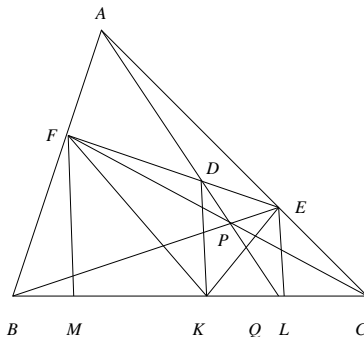
which is the required bound.

---

# Solutions to INMO-2003 problems

1. Consider an acute triangle  $ABC$  and let  $P$  be an interior point of  $ABC$ . Suppose the lines  $BP$  and  $CP$ , when produced, meet  $AC$  and  $AB$  in  $E$  and  $F$  respectively. Let  $D$  be the point where  $AP$  intersects the line segment  $EF$  and  $K$  be the foot of perpendicular from  $D$  on to  $BC$ . Show that  $DK$  bisects  $\angle EKF$ .

**Solution:** Produce  $AP$  to meet  $BC$  in  $Q$ . Join  $KE$  and  $KF$ . Draw perpendiculars from  $F$  and  $E$  on to  $BC$  to meet it in  $M$  and  $L$  respectively. Let us denote  $\angle BKF$  by  $\alpha$  and  $\angle CKE$  by  $\beta$ . We show that  $\alpha = \beta$  by proving  $\tan \alpha = \tan \beta$ . This implies that  $\angle DKF = \angle DKE$ . (See Figure below.)



Since the cevians  $AQ$ ,  $BE$  and  $CF$  concur, we may write

$$\frac{BQ}{QC} = \frac{z}{y}, \frac{CE}{EA} = \frac{x}{z}, \frac{AF}{FB} = \frac{y}{x}.$$

We observe that

$$\frac{FD}{DE} = \frac{[AFD]}{[AED]} = \frac{[PFD]}{[PED]} = \frac{[AFP]}{[AEP]}.$$

However standard computations involving bases give

$$[AFP] = \frac{y}{y+x}[ABP], \quad [AEP] = \frac{z}{z+x}[ACP],$$

and

$$[ABP] = \frac{z}{x+y+z}[ABC], \quad [ACP] = \frac{y}{x+y+z}[ABC].$$

Thus we obtain

$$\frac{FD}{DE} = \frac{x+z}{x+y}.$$

On the other hand

$$\tan \alpha = \frac{FM}{KM} = \frac{FB \sin B}{KM}, \tan \beta = \frac{EL}{KL} = \frac{EC \sin C}{KL}.$$

Using  $FB = \left(\frac{x}{x+y}\right)AB$ ,  $EC = \left(\frac{x}{x+z}\right)AC$  and  $AB \sin B = AC \sin C$ , we obtain

$$\begin{aligned} \frac{\tan \alpha}{\tan \beta} &= \left(\frac{x+z}{x+y}\right) \left(\frac{KL}{KM}\right) \\ &= \left(\frac{x+z}{x+y}\right) \left(\frac{DE}{FD}\right) \\ &= \left(\frac{x+z}{x+y}\right) \left(\frac{x+y}{x+z}\right) = 1. \end{aligned}$$

We conclude that  $\alpha = \beta$ .

2. Find all primes  $p$  and  $q$ , and even numbers  $n > 2$ , satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$

**Solution:** Obviously  $p \neq q$ . We write this in the form

$$p(p^{n-1} + p^{n-2} + \cdots + 1) = q(q+1).$$

If  $q \leq p^{n/2} - 1$ , then  $q < p^{n/2}$  and hence we see that  $q^2 < p^n$ . Thus we obtain

$$q^2 + q < p^n + p^{n/2} < p^n + p^{n-1} + \cdots + p,$$

since  $n > 2$ . It follows that  $q \geq p^{n/2}$ . Since  $n > 2$  and is an even number,  $n/2$  is a natural number larger than 1. This implies that  $q \neq p^{n/2}$  by the given condition that  $q$  is a prime. We conclude that  $q \geq p^{n/2} + 1$ . We may also write the above relation in the form

$$p(p^{n/2} - 1)(p^{n/2} + 1) = (p-1)q(q+1).$$

This shows that  $q$  divides  $(p^{n/2} - 1)(p^{n/2} + 1)$ . But  $q \geq p^{n/2} + 1$  and  $q$  is a prime. Hence the only possibility is  $q = p^{n/2} + 1$ . This gives

$$p(p^{n/2} - 1) = (p-1)(q+1) = (p-1)(p^{n/2} + 2).$$

Simplification leads to  $3p = p^{n/2} + 2$ . This shows that  $p$  divide 2. Thus  $p = 2$  and hence  $q = 5$ ,  $n = 4$ . It is easy to verify that these indeed satisfy the given equation.

3. Show that for every real number  $a$  the equation

$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0 \quad (1)$$

has at least one non-real root and find the sum of all the non-real roots of the equation.

**Solution:** Substituting  $x = y + (1/2)$  in the equation, we obtain the equation in  $y$ :

$$8y^4 + 4y^2 + a - \frac{3}{2} = 0. \quad (2)$$

Using the transformation  $z = y^2$ , we get a quadratic equation in  $z$ :

$$8z^2 + 4z + a - \frac{3}{2} = 0. \quad (3)$$

The discriminant of this equation is  $32(2 - a)$  which is nonnegative if and only if  $a \leq 2$ . For  $a \leq 2$ , we obtain the roots

$$z_1 = \frac{-1 + \sqrt{2(2 - a)}}{4}, \quad z_2 = \frac{-1 - \sqrt{2(2 - a)}}{4}.$$

For getting real  $y$  we need  $z \geq 0$ . Obviously  $z_2 < 0$  and hence it gives only non-real values of  $y$ . But  $z_1 \geq 0$  if and only if  $a \leq \frac{3}{2}$ . In this case we obtain two real values for  $y$  and hence two real roots for the original equation (1). Thus we conclude that there are two real roots and two non-real roots for  $a \leq \frac{3}{2}$  and four non-real roots for  $a > \frac{3}{2}$ . Obviously the sum of all the roots of the equation is 2. For  $a \leq \frac{3}{2}$ , two real roots of (2) are given by  $y_1 = +\sqrt{z_1}$  and  $y_2 = -\sqrt{z_1}$ . Hence the sum of real roots of (1) is given by  $y_1 + \frac{1}{2} + y_2 + \frac{1}{2}$  which reduces to 1. It follows the sum of the non-real roots of (1) for  $a \leq \frac{3}{2}$  is also 1. Thus

$$\text{The sum of nonreal roots} = \begin{cases} 1 & \text{for } a \leq \frac{3}{2} \\ 2 & \text{for } a > \frac{3}{2} \end{cases}$$

4. Find all 7-digit numbers formed by using only the digits 5 and 7, and divisible by both 5 and 7.

**Solution:** Clearly, the last digit must be 5 and we have to determine the remaining 6 digits. For divisibility by 7, it is sufficient to consider the number obtained by replacing 7 by 0; for example 5775755 is divisible by 7 if and only 5005055 is divisible by 7. Each such number is obtained by adding some of the numbers from the set  $\{50, 500, 5000, 50000, 500000, 5000000\}$  along with 5. We look at the remainders of these when divided by 7; they are  $\{1, 3, 2, 6, 4, 5\}$ . Thus it is sufficient to check for those combinations of

remainders which add up to a number of the form  $2 + 7k$ , since the last digit is already 5. These are  $\{2\}$ ,  $\{3, 6\}$ ,  $\{4, 5\}$ ,  $\{2, 3, 4\}$ ,  $\{1, 3, 5\}$ ,  $\{1, 2, 6\}$ ,  $\{2, 3, 5, 6\}$ ,  $\{1, 4, 5, 6\}$  and  $\{1, 2, 3, 4, 6\}$ . These correspond to the numbers 7775775, 7757575, 5577775, 7575575, 5777555, 7755755, 5755575, 5557755, 755555.

5. Let  $ABC$  be a triangle with sides  $a, b, c$ . Consider a triangle  $A_1B_1C_1$  with sides equal to  $a + \frac{b}{2}$ ,  $b + \frac{c}{2}$ ,  $c + \frac{a}{2}$ . Show that

$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$

where  $[XYZ]$  denotes the area of the triangle  $XYZ$ .

**Solution:** It is easy to observe that there is a triangle with sides  $a + \frac{b}{2}$ ,  $b + \frac{c}{2}$ ,  $c + \frac{a}{2}$ . Using Heron's formula, we get

$$16[ABC]^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b),$$

and

$$16[A_1B_1C_1]^2 = \frac{3}{16}(a + b + c)(-a + b + 3c)(-b + c + 3a)(-c + a + 3b).$$

Since  $a, b, c$  are the sides of a triangle, there are positive real numbers  $p, q, r$  such that  $a = q + r$ ,  $b = r + p$ ,  $c = p + q$ . Using these relations we obtain

$$\frac{[ABC]^2}{[A_1B_1C_1]^2} = \frac{16pqr}{3(2p + q)(2q + r)(2r + p)}.$$

Thus it is sufficient to prove that

$$(2p + q)(2q + r)(2r + p) \geq 27pqr,$$

for positive real numbers  $p, q, r$ . Using AM-GM inequality, we get

$$2p + q \geq 3(p^2q)^{1/3}, 2q + r \geq 3(q^2r)^{1/3}, 2r + p \geq 3(r^2p)^{1/3}.$$

Multiplying these, we obtain the desired result. We also observe that equality holds if and only if  $p = q = r$ . This is equivalent to the statement that  $ABC$  is equilateral.

6. In a lottery, tickets are given nine-digit numbers using only the digits 1, 2, 3. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose

the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. Determine, with proof, the colour of the ticket bearing the number 123123123.

**Solution:** The following sequence of moves lead to the colour of the ticket bearing the number 123123123:

Line Number	Ticket Number	Colour	Reason
1	122222222	red	Given
2	222222222	green	Given
3	313113113	blue	Lines 1 & 2
4	231331331	green	Lines 1 & 3
5	331331331	blue	Lines 1 & 2
6	123123123	red	Lines 4 & 5

If 123123123 is reached by some other root, red colour must be obtained even along that root. For if for example 123123123 gets blue from some other root, then the following sequence leads to a contradiction:

Line Number	Ticket Number	Colour	Reason
1	122222222	red	Given
2	123123123	blue	Given
3	231311311	green	Lines 1 & 2
4	211331311	green	Lines 1 & 2
5	332212212	red	Lines 4 & 2
6	113133133	blue	Lines 3 & 5
7	331331331	green	Lines 1 & 2
8	222222222	red	Line 6 & 7

Thus the colour of 222222222 is red contradicting that it is green.

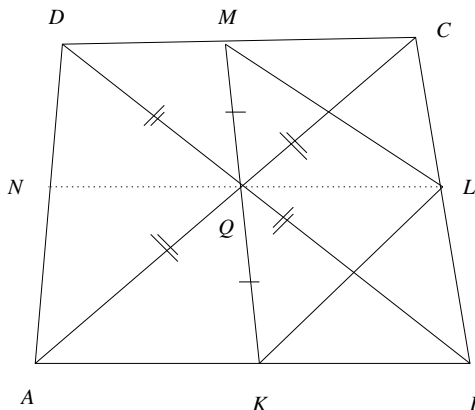
---

# INMO 2004 - Solutions

1. Consider a convex quadrilateral  $ABCD$ , in which  $K, L, M, N$  are the midpoints of the sides  $AB, BC, CD, DA$  respectively. Suppose
  - (a)  $BD$  bisects  $KM$  at  $Q$ ;
  - (b)  $QA = QB = QC = QD$ ; and
  - (c)  $LK/LM = CD/CB$ .

Prove that  $ABCD$  is a **square**.

**Solution:**



**Fig. 1.**

Observe that  $KLMN$  is a parallelogram,  $Q$  is the midpoint of  $MK$  and hence  $NL$  also passes through  $Q$ . Let  $T$  be the point of intersection of  $AC$  and  $BD$ ; and let  $S$  be the point of intersection of  $BD$  and  $MN$ .

Consider the triangle  $MNK$ . Note that  $SQ$  is parallel to  $NK$  and  $Q$  is the midpoint of  $MK$ . Hence  $S$  is the mid-point of  $MN$ . Since  $MN$  is parallel to  $AC$ , it follows that  $T$  is the mid-point of  $AC$ . Now  $Q$  is the circumcentre of  $\triangle ABC$  and the median  $BT$  passes through  $Q$ . Here there are two possibilities:

- (i)  $ABC$  is a right triangle with  $\angle ABC = 90^\circ$  and  $T = Q$ ; and
- (ii)  $T \neq Q$  in which case  $BT$  is perpendicular to  $AC$ .

Suppose  $\angle ABC = 90^\circ$  and  $T = Q$ . Observe that  $Q$  is the circumcentre of the triangle  $DCB$  and hence  $\angle DCB = 90^\circ$ . Similarly  $\angle DAB = 90^\circ$ . It follows that  $\angle ADC = 90^\circ$ . and  $ABCD$  is a rectangle. This implies that  $KLMN$  is a rhombus. Hence  $LK/LM = 1$  and this gives  $CD = CB$ . Thus  $ABCD$  is a square.

In the second case, observe that  $BD$  is perpendicular to  $AC$ ,  $KL$  is parallel to  $AC$  and  $LM$  is parallel to  $BD$ . Hence it follows that  $ML$  is perpendicular to  $LK$ . Similar reasoning shows that  $KLMN$  is a rectangle.

Using  $LK/LM = CD/CB$ , we get that  $CBD$  is similar to  $LMK$ . In particular,  $\angle LMK = \angle CBD = \alpha$  say. Since  $LM$  is parallel to  $DB$ , we also get  $\angle BQK = \alpha$ . Since  $KLMN$  is a cyclic quadrilateral we also get  $\angle LNK = \angle LMK = \alpha$ . Using the fact that  $BD$  is parallel to  $NK$ , we get  $\angle LQB = \angle LNK = \alpha$ . Since  $BD$  bisects  $\angle CBA$ , we also have  $\angle KBQ = \alpha$ . Thus

$$QK = KB = BL = LQ$$

and  $BL$  is parallel to  $QK$ . This gives  $QM$  is parallel to  $LC$  and

$$QM = QL = BL = LC$$

It follows that  $QLCM$  is a parallelogram. But  $\angle LCM = 90^\circ$ . Hence  $\angle MQL = 90^\circ$ . This implies that  $KLMN$  is a square. Also observe that  $\angle LQK = 90^\circ$  and hence  $\angle CBA = \angle LQK = 90^\circ$ . This gives  $\angle CDA = 90^\circ$  and hence  $ABCD$  is a rectangle. Since  $BA = BC$ , it follows that  $ABCD$  is a square.

2. Suppose  $p$  is a prime greater than 3. Find all pairs of integers  $(a, b)$  satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

**Solution:** We write the equation in the form

$$a^2 + 2ap + p^2 + b(3a + 2p) = 0$$

Hence

$$b = \frac{-(a + p)^2}{3a + 2p}$$

is an integer. This shows that  $3a + 2p$  divides  $(a + p)^2$  and hence also divides  $(3a + 3p)^2$ . But, we have

$$(3a + 3p)^2 = (3a + 2p + p)^2 = (3a + 2p)^2 + 2p(3a + 2p) + p^2.$$

It follows that  $3a + 2p$  divides  $p^2$ . Since  $p$  is a prime, the only divisors of  $p^2$  are  $\pm 1, \pm p$  and  $\pm p^2$ . Since  $p > 3$ , we also have  $p = 3k + 1$  or  $3k + 2$ .

**Case 1:** Suppose  $p = 3k + 1$ . Obviously  $3a + 2p = 1$  is not possible. Infact, we get  $1 = 3a + 2p = 3a + 2(3k + 1) \Rightarrow 3a + 6k = -1$  which is impossible. On the other hand  $3a + 2p = -1$  gives  $3a = -2p - 1 = -6k - 3 \Rightarrow a = -2k - 1$  and  $a + p = -2k - 1 + 3k + 1 = k$ .

Thus  $b = \frac{-(a + p)^2}{(3k + 2p)} = k^2$ . Thus  $(a, b) = (-2k - 1, k^2)$  when  $p = 3k + 1$ . Similarly,  $3a + 2p = p \Rightarrow 3a = -p$  which is not possible. Considering  $3a + 2p = -p$ , we get  $3a = -3p$  or  $a = -p \Rightarrow b = 0$ . Hence  $(a, b) = (-3k - 1, 0)$  where  $p = 3k + 1$ .

Let us consider  $3a + 2p = p^2$ . Hence  $3a = p^2 - 2p = p(p - 2)$  and neither  $p$  nor  $p - 2$  is divisible by 3. If  $3a + 2p = -p^2$ , then  $3a = -p(p + 2) \Rightarrow a = -(3k + 1)(k + 1)$ .

Hence  $a + p = (3k + 1)(-k - 1 + 1) = -(3k + 1)k$ . This gives  $b = k^2$ . Again  $(a, b) = \left( -(k + 1)(3k + 1), k^2 \right)$  when  $p = 3k + 1$ .

**Case 2:** Suppose  $p = 3k - 1$ . If  $3a + 2p = 1$ , then  $3a = -6k + 3$  or  $a = -2k + 1$ . We also get

$$b = \frac{-(a + p)^2}{1} = \frac{-(-2k + 1 + 3k - 1)^2}{1} = -k^2$$

and we get the solution  $(a, b) = (-2k + 1, k^2)$ . On the other hand  $3a + 2p = -1$  does not have any solution integral solution for  $a$ . Similarly, there is no solution in the case  $3a + 2p = p$ . Taking  $3a + 2p = -p$ , we get  $a = -p$  and hence  $b = 0$ . We get the solution  $(a, b) = (-3k + 1, 0)$ . If  $3a + 2p = p^2$ , then  $3a = p(p - 2) = (3k - 1)(3k - 3)$  giving  $a = (3k - 1)(k - 1)$  and hence  $a + p = (3k - 1)(1 + k - 1) = k(3k - 1)$ . This gives  $b = -k^2$  and hence  $(a, b) = (3k - 1, -k^2)$ . Finally  $3a + 2p = -p^2$  does not have any solution.

3. If  $\alpha$  is a real root of the equation  $x^5 - x^3 + x - 2 = 0$ , prove that  $[\alpha^6] = 3$ . (For any real number  $a$ , we denote by  $[a]$  the greatest integer not exceeding  $a$ .)

**Solution:** Suppose  $\alpha$  is a real root of the given equation. Then

$$\alpha^5 - \alpha^3 + \alpha - 2 = 0. \quad \dots (1)$$

This gives  $\alpha^5 - \alpha^3 + \alpha - 1 = 1$  and hence  $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) = 1$ . Observe that  $\alpha^4 + \alpha^3 + 1 \geq 2\alpha^2 + \alpha^3 = \alpha^2(\alpha + 2)$ . If  $-1 \leq \alpha < 0$ , then  $\alpha + 2 > 0$ , giving  $\alpha^2(\alpha + 2) > 0$  and hence  $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$ . If  $\alpha < -1$ , then  $\alpha^4 + \alpha^3 = \alpha^3(\alpha + 1) > 0$  and hence  $\alpha^4 + \alpha^3 + 1 > 0$ . This again gives  $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$ .

The above reasoning shows that for  $\alpha < 0$ , we have  $\alpha^5 - \alpha^3 + \alpha - 1 < 0$  and hence cannot be equal to 1. We conclude that a real root  $\alpha$  of  $x^5 - x^3 + x - 2 = 0$  is positive (obviously  $\alpha \neq 0$ ).

Now using  $\alpha^5 - \alpha^3 + \alpha - 2 = 0$ , we get

$$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$$

The statement  $[\alpha^6] = 3$  is equivalent to  $3 \leq \alpha^6 < 4$ .

Consider  $\alpha^4 - \alpha^2 + 2\alpha < 4$ . Since  $\alpha > 0$ , this is equivalent to  $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$ . Using the relation (1), we can write  $2\alpha^2 - \alpha + 2 < 4\alpha$  or  $2\alpha^2 - 5\alpha + 2 < 0$ . Treating this as a quadratic, we get this is equivalent to  $\frac{1}{2} < \alpha < 2$ . Now observe that if  $\alpha \geq 2$  then  $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) \geq 25$  which is impossible. If  $0 < \alpha \leq \frac{1}{2}$ , then  $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$  which again is impossible. We conclude that  $\frac{1}{2} < \alpha < 2$ . Similarly  $\alpha^4 - \alpha^2 + 2\alpha \geq 3$  is equivalent to  $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$  which is equivalent to  $2\alpha^2 - 4\alpha + 2 \geq 0$ . But this is  $2(\alpha - 1)^2 \geq 0$  which is valid. Hence  $3 \leq \alpha^6 < 4$  and we get  $[\alpha^6] = 3$ .

4. Let  $R$  denote the circumradius of a triangle  $ABC$ ;  $a, b, c$  its sides  $BC, CA, AB$ ; and  $r_a, r_b, r_c$  its exradii opposite  $A, B, C$ . If  $2R \leq r_a$ , prove that

- (i)  $a > b$  and  $a > c$ ;
- (ii)  $2R > r_b$  and  $2R > r_c$ .

**Solution:** We know that  $2R = \frac{abc}{2\Delta}$  and  $r_a = \frac{\Delta}{s - a}$ , where  $a, b, c$  are the sides of the triangle  $ABC$ ,  $s = \frac{a + b + c}{2}$  and  $\Delta$  is the area of  $ABC$ . Thus the given condition  $2R \leq r_a$  translates to

$$abc \leq \frac{2\Delta^2}{s - a}$$

Putting  $s - a = p, s - b = q, s - c = r$ , we get  $a = q + r, b = r + p, c = p + q$  and the condition now is

$$p(p + q)(q + r)(r + p) \leq 2\Delta^2$$

But Heron's formula gives,  $\Delta^2 = s(s - a)(s - b)(s - c) = pqr(p + q + r)$ . We obtain  $(p + q)(q + r)(r + p) \leq 2qr(p + q + r)$ . Expanding and effecting some cancellations, we get

$$p^2(q + r) + p(q^2 + r^2) \leq qr(q + r). \quad (\star)$$

Suppose  $a \leq b$ . This implies that  $q + r \leq r + p$  and hence  $q \leq p$ . This implies that  $q^2r \leq p^2r$  and  $qr^2 \leq pr^2$  giving  $qr(q + r) \leq p^2r + pr^2 < p^2r + pr^2 + p^2q + pq^2 = p^2(q + r) + p(q^2 + r^2)$  which contradicts  $(\star)$ . Similarly,  $a \leq c$  is also not possible. This proves (i).

Suppose  $2R \leq r_b$ . As above this takes the form

$$q^2(r + p) + q(r^2 + p^2) \leq pr(p + r). \quad (\star\star)$$

Since  $a > b$  and  $a > c$ , we have  $q > p, r > p$ . Thus  $q^2r > p^2r$  and  $qr^2 > pr^2$ . Hence

$$q^2(r + p) + q(r^2 + p^2) > q^2r + qr^2 > p^2r + pr^2 = pr(p + r)$$

which contradicts  $(\star\star)$ . Hence  $2R > r_b$ . Similarly, we can prove that  $2R > r_c$ . This proves (ii)

5. Let  $S$  denote the set of all 6-tuples  $(a, b, c, d, e, f)$  of positive integers such that  $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$ . Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

Find the greatest common divisor of all the members of  $T$ .

**Solution:** We show that the required gcd is 24. Consider an element  $(a, b, c, d, e, f) \in S$ . We have

$$a^2 + b^2 + c^2 + d^2 + e^2 = f^2.$$

We first observe that not all  $a, b, c, d, e$  can be odd. Otherwise, we have  $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod{8}$  and hence  $f^2 \equiv 5 \pmod{8}$ , which is impossible because no square can be congruent to 5 modulo 8. Thus at least one of  $a, b, c, d, e$  is even.

Similarly if none of  $a, b, c, d, e$  is divisible by 3, then  $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod{3}$  and hence  $f^2 \equiv 2 \pmod{3}$  which again is impossible because no square is congruent to 2 modulo 3. Thus 3 divides  $abcdef$ .

There are several possibilities for  $a, b, c, d, e$ .

**Case 1:** Suppose one of them is even and the other four are odd; say  $a$  is even,  $b, c, d, e$  are odd. Then  $b^2 + c^2 + d^2 + e^2 \equiv 4 \pmod{8}$ . If  $a^2 \equiv 4 \pmod{8}$ , then  $f^2 \equiv 0 \pmod{8}$  and hence  $2|a, 4|f$  giving  $8|af$ . If  $a^2 \equiv 0 \pmod{8}$ , then  $f^2 \equiv 4 \pmod{8}$  which again gives that  $4|a$  and  $2|f$  so that  $8|af$ . It follows that  $8|abcdef$  and hence  $24|abcdef$ .

**Case 2:** Suppose  $a, b$  are even and  $c, d, e$  are odd. Then  $c^2 + d^2 + e^2 \equiv 3 \pmod{8}$ . Since  $a^2 + b^2 \equiv 0$  or  $4 \pmod{8}$ , it follows that  $f^2 \equiv 3$  or  $7 \pmod{8}$  which is impossible. Hence this case does not arise.

**Case 3:** If three of  $a, b, c, d, e$  are even and two odd, then  $8|abcdef$  and hence  $24|abcdef$ .

**Case 4:** If four of  $a, b, c, d, e$  are even, then again  $8|abcdef$  and  $24|abcdef$ . Here again for any six tuple  $(a, b, c, d, e, f)$  in  $S$ , we observe that  $24|abcdef$ . Since

$$1^2 + 1^2 + 1^2 + 2^2 + 3^2 = 4^2.$$

We see that  $(1, 1, 1, 2, 3, 4) \in S$  and hence  $24 \in T$ . Thus 24 is the gcd of  $T$ .

6. Prove that the number of 5-tuples of positive integers  $(a, b, c, d, e)$  satisfying the equation

$$abcde = 5(bcde + acde + abde + abce + abcd)$$

is an **odd** integer.

**Solution:** We write the equation in the form:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{1}{5}.$$

The number of five tuple  $(a, b, c, d, e)$  which satisfy the given relation and for which  $a \neq b$  is even, because for if  $(a, b, c, d, e)$  is a solution, then so is  $(b, a, c, d, e)$  which is distinct from  $(a, b, c, d, e)$ . Similarly the number of five tuples which satisfy the equation and for which  $c \neq d$  is also even. Hence it suffices to count only those five tuples  $(a, b, c, d, e)$  for which  $a = b, c = d$ . Thus the equation reduces to

$$\frac{2}{a} + \frac{2}{c} + \frac{1}{e} = \frac{1}{5}.$$

Here again the tuple  $(a, a, c, c, e)$  for which  $a \neq c$  is even because we can associate different solution  $(c, c, a, a, e)$  to this five tuple. Thus it suffices to consider the equation

$$\frac{4}{a} + \frac{1}{e} = \frac{1}{5},$$

and show that the number of pairs  $(a, e)$  satisfying this equation is odd.

This reduces to

$$ae = 20e + 5a$$

or

$$(a - 20)(e - 5) = 100.$$

But observe that

$$\begin{aligned} 100 &= 1 \times 100 = 2 \times 50 = 4 \times 25 = 5 \times 20 \\ &= 10 \times 10 = 20 \times 5 = 25 \times 4 = 50 \times 2 = 100 \times 1. \end{aligned}$$

Note that no factorisation of 100 as product of two negative numbers yield a positive tuple  $(a, e)$ . Hence we get these 9 solutions. This proves that the total number of five tuples  $(a, b, c, d, e)$  satisfying the given equation is odd.

---

# INMO 2005: Problems and Solutions

1. Let  $M$  be the midpoint of side  $BC$  of a triangle  $ABC$ . Let the median  $AM$  intersect the incircle of  $ABC$  at  $K$  and  $L$ ,  $K$  being nearer to  $A$  than  $L$ . If  $AK=KL=LM$ , prove that the sides of triangle  $ABC$  are in the ratio  $5 : 10 : 13$  in some order.

**Solution:**

Let  $I$  be the incentre of triangle  $ABC$  and  $D$  be its projection on  $BC$ . Observe that  $AB \neq AC$  as  $AB = AC$  implies that  $D = L = M$ . So assume that  $AC > AB$ . Let  $N$  be the projection of  $I$  on  $KL$ . Then the perpendicular  $IN$  from  $I$  to  $KL$  is a bisector of  $KL$  and as  $AK = LM$ , it is a bisector of  $AM$  also. Hence  $AI = IM$ .

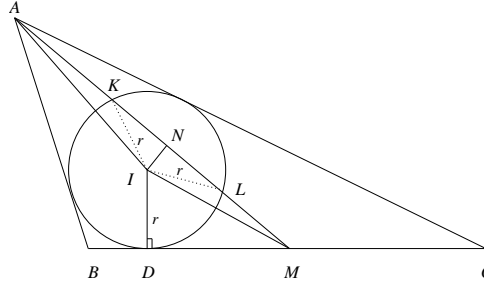


Fig. 1.

But  $AI = \frac{r}{\sin(A/2)} = r \operatorname{cosec}(A/2)$  and

$$\begin{aligned} IM^2 &= ID^2 + DM^2 = r^2 + (BM - BD)^2 \\ &= r^2 + \left(\frac{a}{2} - (s - b)\right)^2. \end{aligned}$$

Hence  $r^2 \operatorname{cosec}^2(A/2) = r^2 + ((a/2) - (s - b))^2$  giving  $r^2 \cot^2(A/2) = ((b - c)/2)^2$ . Since  $b > c$ , we obtain  $r \cot(A/2) = ((b - c)/2)$ . So  $s - a = ((b - c)/2)$ . This gives  $a = 2c$ .

As  $KN = NL$  and  $AK = KL = LM$ , we have  $NL = AM/6$ . We also have  $AN = NM$ . Now

$$\begin{aligned} r^2 = IL^2 = IN^2 + NL^2 &= AI^2 - AN^2 + NL^2 \\ &= AI^2 - \frac{1}{4}m_a^2 + \frac{1}{36}m_a^2 \\ &= r^2 \operatorname{cosec}^2(A/2) - \frac{2}{9}m_a^2. \end{aligned}$$

Hence  $r^2 \cot^2(A/2) = \frac{2}{9}m_a^2$ . From the above, we get

$$\left(\frac{b - c}{2}\right)^2 = \frac{2}{9} \cdot \frac{1}{4}(2b^2 + 2c^2 - a^2).$$

Simplification gives  $5b^2 + 13c^2 - 18bc = 0$ . This can be written as  $(b - c)(5b - 13c) = 0$ . As  $b \neq c$ , we get  $5b - 13c = 0$ . To conclude,  $a = 2c, 5b = 13c$  yield

$$\frac{a}{10} = \frac{b}{13} = \frac{c}{5}.$$

2. Let  $\alpha$  and  $\beta$  be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of  $\beta$ .

**Solution:**

We have

$$\frac{77}{17} < \frac{\beta}{\alpha} < \frac{197}{43}.$$

That is,

$$4 + \frac{9}{17} < \frac{\beta}{\alpha} < 4 + \frac{25}{43}.$$

Thus  $4 < \frac{\beta}{\alpha} < 5$ . Since  $\alpha$  and  $\beta$  are positive integers, we may write  $\beta = 4\alpha + x$ , where  $0 < x < \alpha$ . Now we get

$$4 + \frac{9}{17} < 4 + \frac{x}{\alpha} < 4 + \frac{25}{43}.$$

So  $\frac{9}{17} < \frac{x}{\alpha} < \frac{25}{43}$ ; that is,  $\frac{43x}{25} < \alpha < \frac{17x}{9}$ .

We find the smallest value of  $x$  for which  $\alpha$  becomes a well-defined integer. For  $x = 1, 2, 3$  the bounds of  $\alpha$  are respectively  $\left(1\frac{18}{25}, 1\frac{8}{9}\right)$ ,  $\left(3\frac{11}{25}, 3\frac{7}{9}\right)$ ,  $\left(5\frac{4}{9}, 5\frac{2}{3}\right)$ . None of these pairs contain an integer between them.

For  $x = 4$ , we have  $\frac{43x}{25} = 6\frac{12}{25}$  and  $\frac{17x}{9} = 7\frac{5}{9}$ . Hence, in this case  $\alpha = 7$ , and  $\beta = 4\alpha + x = 28 + 4 = 32$ .

This is also the least possible value, because, if  $x \geq 5$ , then  $\alpha > \frac{43x}{25} \geq \frac{43}{5} > 8$ , and so  $\beta > 37$ . Hence the minimum possible value of  $\beta$  is 32.

3. Let  $p, q, r$  be positive real numbers, not all equal, such that some two of the equations

$$px^2 + 2qx + r = 0, \quad qx^2 + 2rx + p = 0, \quad rx^2 + 2px + q = 0,$$

have a common root, say  $\alpha$ . Prove that

- (a)  $\alpha$  is real and negative; and
- (b) the third equation has non-real roots.

**Solution:**

Consider the discriminants of the three equations

$$px^2 + qx + r = 0 \tag{1}$$

$$qx^2 + rx + p = 0 \tag{2}$$

$$rx^2 + px + q = 0. \tag{3}$$

Let us denote them by  $D_1, D_2, D_3$  respectively. Then we have

$$D_1 = 4(q^2 - rp), D_2 = 4(r^2 - pq), D_3 = 4(p^2 - qr).$$

We observe that

$$\begin{aligned} D_1 + D_2 + D_3 &= 4(p^2 + q^2 + r^2 - pq - qr - rp) \\ &= 2\{(p - q)^2 + (q - r)^2 + (r - p)^2\} > 0 \end{aligned}$$

since  $p, q, r$  are not all equal. Hence at least one of  $D_1, D_2, D_3$  must be positive. We may assume  $D_1 > 0$ .

Suppose  $D_2 < 0$  and  $D_3 < 0$ . In this case both the equations (2) and (3) have only non-real roots and equation (1) has only real roots. Hence the common root  $\alpha$  must be between (2) and (3). But then  $\bar{\alpha}$  is the other root of both (2) and (3). Hence it follows that (2) and (3) have same set of roots. This implies that

$$\frac{q}{r} = \frac{r}{p} = \frac{p}{q}.$$

Thus  $p = q = r$  contradicting the given condition. Hence both  $D_2$  and  $D_3$  cannot be negative. We may assume  $D_2 \geq 0$ . Thus we have

$$q^2 - rp > 0, r^2 - pq \geq 0.$$

These two give

$$q^2 r^2 > p^2 qr$$

since  $p, q, r$  are all positive. Hence we obtain  $qr > p^2$  or  $D_3 < 0$ . We conclude that the common root must be between equations (1) and (2).

Thus

$$\begin{aligned} p\alpha^2 + q\alpha + r &= 0 \\ q\alpha^2 + r\alpha + p &= 0 \end{aligned}$$

Eliminating  $\alpha^2$ , we obtain

$$2(q^2 - pr)\alpha = p^2 - qr.$$

Since  $q^2 - pr > 0$  and  $p^2 - qr < 0$ , we conclude that  $\alpha < 0$ .

The condition  $p^2 - qr < 0$  implies that the equation (3) has only non-real roots.

Alternately one can argue as follows. Suppose  $\alpha$  is a common root of two equations, say, (1) and (2). If  $\alpha$  is non-real, then  $\bar{\alpha}$  is also a root of both (1) and (2). Hence The coefficients of (1) and (2) are proportional. This forces  $p = q = r$ , a contradiction. Hence the common root between any two equations cannot be non-real. Looking at the coefficients, we conclude that the common root  $\alpha$  must be negative. If (1) and (2) have common root  $\alpha$ , then  $q^2 \geq rp$  and  $r^2 \geq pq$ . Here at least one inequality is strict for  $q^2 = pr$  and  $r^2 = pq$  forces  $p = q = r$ . Hence  $q^2 r^2 > p^2 qr$ . This gives  $p^2 < qr$  and hence (3) has nonreal roots.

4. All possible 6-digit numbers, in each of which the digits occur in **non-increasing** order (from left to right, e.g., 877550) are written as a sequence in **increasing** order. Find the 2005-th number in this sequence.

### Solution I:

Consider a 6-digit number whose digits from left to right are in non increasing order. If 1 is the first digit of such a number, then the subsequent digits cannot exceed 1. The set of all such numbers with initial digit equal to 1 is

$$\{100000, 110000, 111000, 111100, 111110, 111111\}.$$

There are elements in this set.

Let us consider 6-digit numbers with initial digit 2. Starting from 200000, we can go up to 222222. We count these numbers as follows:

$$\begin{array}{rclcl} 200000 & - & 211111 & : & 6 \\ 220000 & - & 221111 & : & 5 \\ 222000 & - & 222111 & : & 4 \\ 222200 & - & 222211 & : & 3 \\ 222220 & - & 222221 & : & 2 \\ 222222 & - & 222222 & : & 1 \end{array}$$

The number of such numbers is 21. Similarly we count numbers with initial digit 3; the sequence starts from 300000 and ends with 333333. We have

300000	-	322222	:	21
330000	-	332222	:	15
333000	-	333222	:	10
333300	-	333322	:	6
333330	-	333332	:	3
333333	-	333333	:	1

We obtain the total number of numbers starting from 3 equal to 56. Similarly,

400000	-	433333	:	56
440000	-	443333	:	35
444000	-	444333	:	20
444400	-	444433	:	10
444440	-	444443	:	4
444444	-	444444	:	1
				<u>126</u>

500000	-	544444	:	126
550000	-	554444	:	70
555000	-	555444	:	35
555500	-	555544	:	15
555550	-	555554	:	5
555555	-	555555	:	1
				<u>252</u>

600000	-	655555	:	252
660000	-	665555	:	126
666000	-	666555	:	56
666600	-	666655	:	21
666660	-	666665	:	6
666666	-	666666	:	1
				<u>462</u>

700000	-	766666	:	462
770000	-	776666	:	210
777000	-	777666	:	84
777700	-	777766	:	28
777770	-	777776	:	7
777777	-	777777	:	1
				<u>792</u>

Thus the number of 6-digit numbers where digits are non-increasing starting from 100000 and ending with 777777 is

$$792 + 462 + 252 + 126 + 56 + 21 + 6 = 1715.$$

Since  $2005 - 1715 = 290$ , we have to consider only 290 numbers in the sequence with initial digit 8. We have

800000	-	855555	:	252
860000	-	863333	:	35
864000	-	864110	:	3

Thus the required number is 864110.

**Solution: II**

It is known that the number of ways of choosing  $r$  objects from  $n$  different types of objects (with repetitions allowed) is  $\binom{n+r-1}{r}$ . In particular, if we want to write  $r$ -digit numbers using  $n$  digits allowing for repetitions with the additional condition that the digits appear in non-increasing order, we see that this can be done in  $\binom{n+r-1}{r}$  ways.

Now we group the given numbers into different classes and write the number of ways in which each class can be obtained. To keep track we also write the cumulative sums of the number of numbers so obtained. Observe that the numbers themselves are written in ascending order. So we exhaust numbers beginning with 1, then beginning with 2 and so on.

Numbers	Digits used other than the fixed part	$n$	$r$	$\binom{n+r-1}{r}$	Cumulative sum
beginning with 1	1,0	2	5	$\binom{6}{5} = 6$	6
2	2,1,0	3	5	$\binom{7}{5} = 21$	27
3	3,2,1,0	4	5	$\binom{8}{5} = 56$	83
4	4,3,2,1,0	5	5	$\binom{9}{5} = 126$	209
5	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	461
6	6,5,4,3,2,1,0	7	5	$\binom{11}{5} = 462$	923
7	7,6,5,4,3,2,1,0	8	5	$\binom{12}{5} = 792$	1715
from 800000 to 855555	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	1967
from 860000 to 863333	3,2,1,0	4	4	$\binom{7}{4} = 35$	2002

The next three 6-digit numbers are 864000, 864100, 864110.

Hence the 2005th number in the sequence is 864110.

5. Let  $x_1$  be a given positive integer. A sequence  $\langle x_n \rangle_{n=1}^{\infty} = \langle x_1, x_2, x_3, \dots \rangle$  of positive integers is such that  $x_n$ , for  $n \geq 2$ , is obtained from  $x_{n-1}$  by adding some nonzero digit of  $x_{n-1}$ . Prove that
- (a) the sequence has an **even** number;
  - (b) the sequence has infinitely many even numbers.

**Solution:**

- (a) Let us assume that there are no even numbers in the sequence. This means that  $x_{n+1}$  is obtained from  $x_n$ , by adding a nonzero even digit of  $x_n$  to  $x_n$ , for each  $n \geq 1$ . Let  $E$  be the left most even digit in  $x_1$  which may be taken in the form

$$x_1 = O_1 O_2 \dots O_k E D_1 D_2 \dots D_l$$

where  $O_1, O_2, \dots, O_k$  are odd digits ( $k \geq 0$ );  $D_1, D_2, \dots, D_{l-1}$  are even or odd; and  $D_l$  odd,  $l \geq 1$ .

Since each time we are adding at least 2 to a term of the sequence to get the next term, at some stage, we will have a term of the form

$$x_r = O_1 O_2 \dots O_k E 999 \dots 9 F$$

where  $F = 3, 5, 7$  or  $9$ . Now we are forced to add  $E$  to  $x_r$  to get  $x_{r+1}$ , as it is the only even digit available. After at most four steps of addition, we see that some next term is of the form

$$x_s = O_1 O_2 \dots O_k G 000 \dots M$$

where  $G$  replaces  $E$  of  $x_r$ ,  $G = E + 1$ ,  $M = 1, 3, 5$ , or  $7$ . But  $x_s$  has no nonzero even digit contradicting our assumption. Hence the sequence has some even number as its term.

- (b) If there are only finitely many even terms and  $x_t$  is the last term, then the sequence  $\langle x_n \rangle_{n=t+1}^\infty = \langle x_{t+1}, x_{t+2}, \dots \rangle$  is obtained in a similar manner and hence must have an even term by (a), a contradiction. Thus  $\langle x_n \rangle_{n=1}^\infty$  has infinitely many even terms.

6. Find all functions  $f : \mathbf{R} \rightarrow \mathbf{R}$  such that

$$f(x^2 + yf(z)) = xf(x) + zf(y) \quad (1)$$

for all  $x, y, z$  in  $\mathbf{R}$ . (Here  $\mathbf{R}$  denotes the set of all real numbers.)

**Solution:** Taking  $x = y = 0$  in (1), we get  $zf(0) = f(0)$  for all  $z \in \mathbf{R}$ . Hence we obtain  $f(0) = 0$ . Taking  $y = 0$  in (1), we get

$$f(x^2) = xf(x) \quad (2)$$

Similarly  $x = 0$  in (1) gives

$$f(yf(z)) = zf(y) \quad (3)$$

Putting  $y = 1$  in (3), we get

$$f(f(z)) = zf(1) \quad \forall z \in \mathbf{R} \quad (4)$$

Now using (2) and (4), we obtain

$$f(xf(x)) = f(f(x^2)) = x^2 f(1) \quad (5)$$

Put  $y = z = x$  in (3) also given

$$f(xf(x)) = xf(x) \quad (6)$$

Comparing (5) and (6), it follows that  $x^2 f(1) = xf(x)$ . If  $x \neq 0$ , then  $f(x) = cx$ , for some constant  $c$ . Since  $f(0) = 0$ , we have  $f(x) = cx$  for  $x = 0$  as well. Substituting this in (1), we see that

$$c(x^2 + cyz) = cx^2 + cyz$$

or

$$c^2 yz = cyz \quad \forall y, z \in \mathbf{R}.$$

This implies that  $c^2 = c$ . Hence  $c = 0$  or  $1$ . We obtain  $f(x) = 0$  for all  $x$  or  $f(x) = x$  for all  $x$ . It is easy to verify that these two are solutions of the given equation.

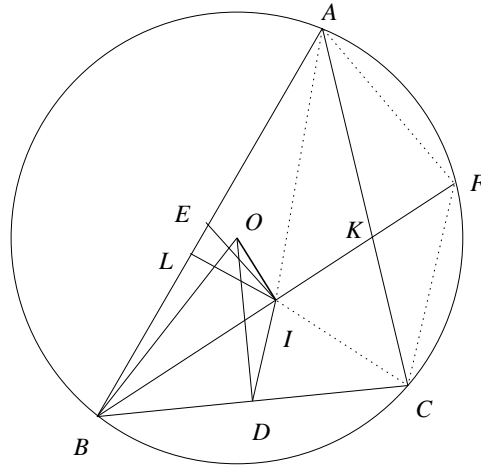
————— × × × —————

# INMO 2006: Problems and Solutions

1. In a non-equilateral triangle  $ABC$ , the sides  $a, b, c$  form an arithmetic progression. Let  $I$  and  $O$  denote the incentre and circumcentre of the triangle respectively.
  - (i) Prove that  $IO$  is perpendicular to  $BI$ .
  - (ii) Suppose  $BI$  extended meets  $AC$  in  $K$ , and  $D, E$  are the midpoints of  $BC, BA$  respectively. Prove that  $I$  is the circumcentre of triangle  $DKE$ .

**Solution:**

- (i) Extend  $BI$  to meet the circumcircle in  $F$ . Then we know that  $FA = FI = FC$ . (See Figure)



Let  $BI : IF = \lambda : \mu$ . Applying Stewart's theorem to triangle  $BAF$ , we get

$$\lambda AF^2 + \mu AB^2 = (\lambda + \mu)(AI^2 + BI \cdot IF).$$

Similarly, Stewart's theorem to triangle  $BCF$  gives

$$\lambda CF^2 + \mu BC^2 = (\lambda + \mu)(CI^2 + BI \cdot IF).$$

Since  $CF = AF$ , subtraction gives

$$\mu(AB^2 - BC^2) = (\lambda + \mu)(AI^2 - CI^2).$$

Using the standard notations  $AB = c$ ,  $BC = a$ ,  $CA = b$  and  $s = (a + b + c)/2$ , we get  $AI^2 = r^2 + (s - a)^2$  and  $CI^2 = r^2 + (s - c)^2$  where  $r$  is the in-radius of  $ABC$ . Thus

$$\mu(c^2 - a^2) = (\lambda + \mu)((s - a)^2 - (s - c)^2) = (\lambda + \mu)(c - a)b.$$

It follows that either  $c = a$  or  $\mu(c + a) = (\lambda + \mu)b$ . But  $c = a$  implies that  $a = b = c$  since  $a, b, c$  are in arithmetic progression. However, we have taken a non-equilateral triangle  $ABC$ . Thus  $c \neq a$  and we have  $\mu(c + a) = (\lambda + \mu)b$ . But  $c + a = 2b$  and we obtain

$2b\mu = (\lambda + \mu)b$ . We conclude that  $\lambda = \mu$ . This in turn tells that  $I$  is the mid-point of  $BF$ . Since  $OF = OB$ , we conclude that  $OI$  is perpendicular to  $BF$ .

### Alternatively

Applying Ptolemy's theorem to the cyclic quadrilateral  $ABCF$ , we get

$$AB \cdot CF + AF \cdot BC = BF \cdot CA.$$

Since  $CF = AF$ , we get  $CF(c+a) = BF \cdot b = BF(c+a)/2$ . This gives  $BF = 2CF = 2IF$ . Hence  $I$  is the mid-point of  $BF$  and as earlier we conclude that  $OI$  is perpendicular to  $BF$ .

### Alternatively

Join  $BO$ . We have to prove that  $\angle BIO = 90^\circ$ , which is equivalent to  $BI^2 + IO^2 = BO^2$ . Draw  $IL$  perpendicular to  $AB$ . Let  $R$  denote the circumradius of  $ABC$  and let  $\Delta$  denote its area. Observe that  $BO = R$ ,  $IO^2 = R^2 - 2Rr$ ,

$$BI = \frac{BL}{\cos(B/2)} = (s-b)\sqrt{\frac{ca}{s(s-b)}}.$$

Thus we obtain

$$BI^2 = ac(s-b)/s = \frac{ac}{3},$$

since  $a, b, c$  are in arithmetic progression. Thus we need to prove that

$$\frac{ac}{3} + R^2 - 2Rr = R^2.$$

This reduces to proving  $2Rr = ac/3$ . But

$$2Rr = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{abc}{2s} = \frac{abc}{a+b+c} = \frac{ac}{3},$$

using  $a + c = 2b$ . This proves the claim.

- (ii) Join  $ID$ . Note that  $\angle BIO = \angle BDO = 90^\circ$ . Hence  $B, D, I, O$  are concyclic and hence  $\angle BID = \angle BOD = A$ . Since  $\angle DBI = \angle KBA = B/2$ , it follows that triangles  $BAK$  and  $BID$  are similar. This gives

$$\frac{BA}{BI} = \frac{BK}{BD} = \frac{AK}{ID}.$$

However, we have seen earlier that  $BI = ac/3$ . Moreover  $AK = bc/(a+c)$ . Thus we obtain

$$BK = \frac{BA \cdot BD}{BI} = \frac{1}{2}\sqrt{3ac}, \quad ID = \frac{AK \cdot BI}{BA} = \frac{1}{2}\sqrt{\frac{ac}{3}}.$$

By symmetry, we must have  $IE = \frac{1}{2}\sqrt{\frac{ac}{3}}$ . Finally

$$IK = \frac{b}{a+b+c} \cdot BK = \frac{1}{3}BK = \frac{1}{2}\sqrt{\frac{ac}{3}}.$$

Thus  $ID = IE = IK$  and  $I$  is the circumcentre of  $DKE$ .

### Alternatively

Observe that  $AK = bc/(a+c) = c/2 = AE$ . Since  $AI$  bisects angle  $A$ , we see that  $AIE$  is congruent to  $AIK$ . This gives  $IE = IK$ . Similarly  $CID$  is congruent to  $CIK$  giving  $ID = IK$ . We conclude that  $ID = IK = IE$ .

2. Prove that for every positive integer  $n$  there exists a **unique** ordered pair  $(a, b)$  of positive integers such that

$$n = \frac{1}{2}(a + b - 1)(a + b - 2) + a.$$

**Solution:** We have to prove that  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  defined by

$$f(a, b) = \frac{1}{2}(a + b - 1)(a + b - 2) + a, \quad \forall a, b \in \mathbb{N},$$

is a bijection. (Note that the right side is a natural number.) To this end define

$$T(n) = \frac{n(n+1)}{2}, \quad n \in \mathbb{N} \cup \{0\}.$$

An idea of the proof can be obtained by looking at the following table of values of  $f(a, b)$  for some small values of  $a, b$ .

$b \backslash a$	1	2	3	4	5	6
1	1	2	4	7	11	16
2	3	5	8	12	17	
3	6	9	13	18		
4	10	14	19			
5	15	20				
6	21					

We observe that the  $n$ -th diagonal runs from  $(1, n)$ -th position to  $(n, 1)$ -th position and the entries are  $n$  consecutive integers; the first entry in the  $n$ -th diagonal is one more than the last entry of the  $(n - 1)$ -th diagonal. For example the first entry in 5-th diagonal is 11 which is one more than the last entry of 4-th diagonal which is 10. Observe that 5-th diagonal starts from 11 and ends with 15 which accounts for 5 consecutive natural numbers. Thus we see that  $f(n - 1, 1) + 1 = f(1, n)$ . We also observe that the first  $n$  diagonals exhaust all the natural numbers from 1 to  $T(n)$ . (Thus a kind of visual bijection is already there. We formally prove the property.)

We first observe that

$$f(a, b) - T(a + b - 2) = a > 0,$$

and

$$T(a + b - 1) - f(a, b) = \frac{(a + b - 1)(a + b)}{2} - \frac{(a + b - 1)(a + b - 2)}{2} - a = b - 1 \geq 0.$$

Thus we have

$$T(a+b-2) < f(a,b) = \frac{(a+b-1)(a+b-2)}{2} + a \leq T(a+b-1).$$

Suppose  $f(a_1, b_1) = f(a_2, b_2)$ . Then the previous observation shows that

$$\begin{aligned} T(a_1 + b_1 - 2) &< f(a_1, b_1) \leq T(a_1 + b_1 - 1), \\ T(a_2 + b_2 - 2) &< f(a_2, b_2) \leq T(a_2 + b_2 - 1). \end{aligned}$$

Since the sequence  $\langle T(n) \rangle_{n=0}^\infty$  is strictly increasing, it follows that  $a_1 + b_1 = a_2 + b_2$ . But then the relation  $f(a_1, b_1) = f(a_2, b_2)$  implies that  $a_1 = a_2$  and  $b_1 = b_2$ . Hence  $f$  is one-one.

Let  $n$  be any natural number. Since the sequence  $\langle T(n) \rangle_{n=0}^\infty$  is strictly increasing, we can find a natural number  $k$  such that

$$T(k-1) < n \leq T(k).$$

Equivalently,

$$\frac{(k-1)k}{2} < n \leq \frac{k(k+1)}{2}. \quad (1)$$

Now set  $a = n - \frac{k(k-1)}{2}$  and  $b = k - a + 1$ . Observe that  $a > 0$ . Now (1) shows that

$$a = n - \frac{k(k-1)}{2} \leq \frac{k(k+1)}{2} - \frac{k(k-1)}{2} = k.$$

Hence  $b = k - a + 1 \geq 1$ . Thus  $a$  and  $b$  are both positive integers and

$$f(a,b) = \frac{1}{2}(a+b-1)(a+b-2) + a = \frac{k(k-1)}{2} + a = n.$$

This shows that every natural number is in the range of  $f$ . Thus  $f$  is also onto. We conclude that  $f$  is a bijection.

3. Let  $X$  denote the set of all triples  $(a, b, c)$  of integers. Define a function  $f : X \rightarrow X$  by

$$f(a, b, c) = (a + b + c, ab + bc + ca, abc).$$

Find all triples  $(a, b, c)$  in  $X$  such that  $f(f(a, b, c)) = (a, b, c)$ .

**Solution:** We show that the solutionset consists of  $\{(t, 0, 0) ; t \in \mathbb{Z}\} \cup \{(-1, -1, 1)\}$ . Let us put  $a + b + c = d$ ,  $ab + bc + ca = e$  and  $abc = f$ . The given condition  $f(f(a, b, c)) = (a, b, c)$  implies that

$$d + e + f = a, \quad de + ef + fd = b, \quad def = c.$$

Thus  $abcdef = fc$  and hence either  $cf = 0$  or  $abde = 1$ .

**Case I:** Suppose  $cf = 0$ . Then either  $c = 0$  or  $f = 0$ . However  $c = 0$  implies  $f = 0$  and vice-versa. Thus we obtain  $a + b = d$ ,  $d + e = a$ ,  $ab = e$  and  $de = b$ . The first two relations give  $b = -e$ . Thus  $e = ab = -ae$  and  $de = b = -e$ . We get either  $e = 0$  or  $a = d = -1$ .

If  $e = 0$ , then  $b = 0$  and  $a = d = t$ , say. We get the triple  $(a, b, c) = (t, 0, 0)$ , where  $t \in \mathbb{Z}$ . If  $e \neq 0$ , then  $a = d = -1$ . But then  $d + e + f = a$  implies that  $-1 + e + 0 = -1$  forcing  $e = 0$ . Thus we get the solution family  $(a, b, c) = (t, 0, 0)$ , where  $t \in \mathbb{Z}$ .

**Case II:** Suppose  $cf \neq 0$ . In this case  $abde = 1$ . Hence either all are equal to 1; or two equal to 1 and the other two equal to  $-1$ ; or all equal to  $-1$ .

Suppose  $a = b = d = e = 1$ . Then  $a + b + c = d$  shows that  $c = -1$ . Similarly  $f = -1$ . Hence  $e = ab + bc + ca = 1 - 1 - 1 = -1$  contradicting  $e = 1$ .

Suppose  $a = b = 1$  and  $d = e = -1$ . Then  $a + b + c = d$  gives  $c = -3$  and  $d + e + f = a$  gives  $f = 3$ . But then  $f = abc = 1 \cdot 1 \cdot (-3) = -3$ , a contradiction. Similarly  $a = b = -1$  and  $d = e = 1$  is not possible.

If  $a = 1, b = -1, d = 1, e = -1$ , then  $a + b + c = d$  gives  $c = 1$ . Similarly  $f = 1$ . But then  $f = abc = 1 \cdot 1 \cdot (-1) = -1$  a contradiction. If  $a = 1, b = -1, d = -1, e = 1$ , then  $c = -1$  and  $e = ab + bc + ca = -1 + 1 - 1 = -1$  and a contradiction to  $e = 1$ . The symmetry between  $(a, b, c)$  and  $(d, e, f)$  shows that  $a = -1, b = 1, d = 1, e = -1$  is not possible. Finally if  $a = -1, b = 1, d = -1$  and  $e = 1$ , then  $c = -1$  and  $f = -1$ . But then  $f = abc$  is not satisfied.

The only case left is that of  $a, b, d, e$  being all equal to  $-1$ . Then  $c = 1$  and  $f = 1$ . It is easy to check that  $(-1, -1, 1)$  is indeed a solution.

### Alternatively

$cf \neq 0$  implies that  $|c| \geq 1$  and  $|f| \geq 1$ . Observe that

$$d^2 - 2e = a^2 + b^2 + c^2, \quad a^2 - 2b = d^2 + e^2 + f^2.$$

Adding these two, we get  $-2(b + e) = b^2 + c^2 + e^2 + f^2$ . This may be written in the form

$$(b + 1)^2 + (e + 1)^2 + c^2 + f^2 - 2 = 0.$$

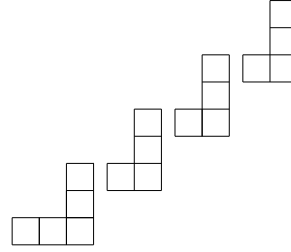
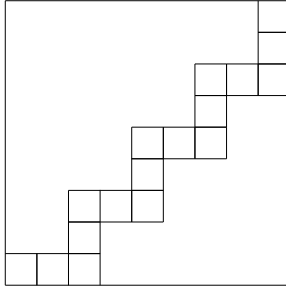
We conclude that  $c^2 + f^2 \leq 2$ . Using  $|c| \geq 1$  and  $|f| \geq 1$ , we obtain  $|c| = 1$  and  $|f| = 1$ ,  $b + 1 = 0$  and  $e + 1 = 0$ . Thus  $b = e = -1$ . Now  $a + d = d + e + f + a + b + c$  and this gives  $b + c + e + f = 0$ . It follows that  $c = f = 1$  and finally  $a = d = -1$ .

4. Some 46 squares are randomly chosen from a  $9 \times 9$  chess board and are coloured red. Show that there exists a  $2 \times 2$  block of 4 squares of which at least three are coloured red.

**Solution:** Consider a partition of  $9 \times 9$  chess board using sixteen  $2 \times 2$  block of 4 squares each and remaining seventeen single squares as shown in the figure below.

1	2	3	4	
7	6	5		
8	9			16
			15	14
10				
		11	12	13

If any one of these 16 big squares contain 3 red squares then we are done. On the contrary, each may contain at most 2 red squares and these account for at most  $16 \cdot 2 = 32$  red squares. Then there are 17 single squares connected in zig-zag fashion. It looks as follows:



We split this again in to several mirror images of L-shaped figures as shown above. There are four such forks. If all the five unit squares of the first fork are red, then we can get a  $2 \times 2$  square having three red squares. Hence there can be at most four unit squares having red colour. Similarly, there can be at most three red squares from each of the remaining three forks. Together we get  $4 + 3 \cdot 3 = 13$  red squares. These together with 32 from the big squares account for only 45 red squares. But we know that 46 squares have red colour. The conclusion follows.

5. In a cyclic quadrilateral  $ABCD$ ,  $AB = a$ ,  $BC = b$ ,  $CD = c$ ,  $\angle ABC = 120^\circ$ , and  $\angle ABD = 30^\circ$ . Prove that

- (i)  $c \geq a + b$ ;
- (ii)  $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$ .

**Solution:**

Applying cosine rule to triangle  $ABC$ , we get

$$AC^2 = a^2 + b^2 - 2ab \cos 120^\circ = a^2 + b^2 + ab.$$

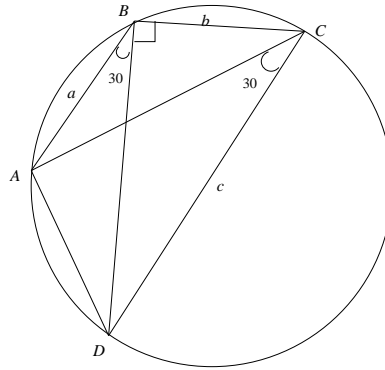
Observe that  $\angle DAC = \angle DBC = 120^\circ - 30^\circ = 90^\circ$ . Thus we get

$$c^2 = \frac{AC^2}{\cos^2 30^\circ} = \frac{4}{3}(a^2 + b^2 + ab).$$

So

$$c^2 - (a+b)^2 = \frac{4}{3}(a^2 + b^2 + ab) - (a^2 + b^2 + 2ab) = \frac{(a-b)^2}{3} \geq 0.$$

This proves  $c \geq a + b$  and thus (i) is true.



For proving (ii), consider the product

$$Q = (\alpha + \beta + \gamma)(\alpha - \beta - \gamma)(\alpha + \beta - \gamma)(\alpha - \beta + \gamma),$$

where  $\alpha = \sqrt{c+a}$ ,  $\beta = \sqrt{c+b}$  and  $\gamma = \sqrt{c-a-b}$ . Expanding the product, we get

$$\begin{aligned} Q &= (c+a)^2 + (c+b)^2 + (c-a-b)^2 - 2(c+a)(c+b) - 2(c+a)(c-a-b) - 2(c+b)(c-a-b) \\ &= -3c^2 + 4a^2 + 4b^2 + 4ab \\ &= 0. \end{aligned}$$

Thus at least one of the factors must be equal to 0. Since  $\alpha + \beta + \gamma > 0$  and  $\alpha + \beta - \gamma > 0$ , it follows that the product of the remaining two factors is 0. This gives

$$\sqrt{c+a} - \sqrt{c+b} = \sqrt{c-a-b} \text{ or } \sqrt{c+a} - \sqrt{c+b} = -\sqrt{c-a-b}.$$

We conclude that

$$|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}.$$

6. (a) Prove that if  $n$  is a positive integer such that  $n \geq 4011^2$ , then there exists an integer  $l$  such that  $n < l^2 < \left(1 + \frac{1}{2005}\right)n$ .
- (b) Find the smallest positive integer  $M$  for which whenever an integer  $n$  is such that  $n \geq M$ , there exists an integer  $l$ , such that  $n < l^2 < \left(1 + \frac{1}{2005}\right)n$ .

**Solution:**

- (a) Let  $n \geq 4011^2$  and  $m \in \mathbb{N}$  be such that  $m^2 \leq n < (m+1)^2$ . Then

$$\begin{aligned} \left(1 + \frac{1}{2005}\right)n - (m+1)^2 &\geq \left(1 + \frac{1}{2005}\right)m^2 - (m+1)^2 \\ &= \frac{m^2}{2005} - 2m - 1 \\ &= \frac{1}{2005}(m^2 - 4010m - 2005) \\ &= \frac{1}{2005}\left((m - 2005)^2 - 2005^2 - 2005\right) \\ &\geq \frac{1}{2005}\left((4011 - 2005)^2 - 2005^2 - 2005\right) \\ &= \frac{1}{2005}\left(2006^2 - 2005^2 - 2005\right) \\ &= \frac{1}{2005}(4011 - 2005) = \frac{2006}{2005} > 0. \end{aligned}$$

Thus we get

$$n < (m+1)^2 < \left(1 + \frac{1}{2005}\right)n,$$

and  $l^2 = (m+1)^2$  is the desired square.

- (b) We show that  $M = 4010^2 + 1$  is the required least number. Suppose  $n \geq M$ . Write  $n = 4010^2 + k$ , where  $k$  is a positive integer. Note that we may assume  $n < 4011^2$  by part (a). Now

$$\begin{aligned}
 \left(1 + \frac{1}{2005}\right)n - 4011^2 &= \left(1 + \frac{1}{2005}\right)(4010^2 + k) - 4011^2 \\
 &= 4010^2 + 2 \cdot 4010 + k + \frac{k}{2005} - 4011^2 \\
 &= (4010 + 1)^2 + (k - 1) + \frac{k}{2005} - 4011^2 \\
 &= (k - 1) + \frac{k}{2005} > 0.
 \end{aligned}$$

Thus we obtain

$$4010^2 < n < 4011^2 < \left(1 + \frac{1}{2005}\right)n.$$

We check that  $M = 4010^2$  will not work. For suppose  $n = 4010^2$ . Then

$$\left(1 + \frac{1}{2005}\right)4010^2 = 4010^2 + 2 \cdot 4010 = 4011^2 - 1 < 4011^2.$$

Thus there is no square integer between  $n$  and  $\left(1 + \frac{1}{2005}\right)n$ .

This proves (b).

—————× × ×—————

# Problems and Solutions of INMO-2007

1. In a triangle  $ABC$  right-angled at  $C$ , the median through  $B$  bisects the angle between  $BA$  and the bisector of  $\angle B$ . Prove that

$$\frac{5}{2} < \frac{AB}{BC} < 3.$$

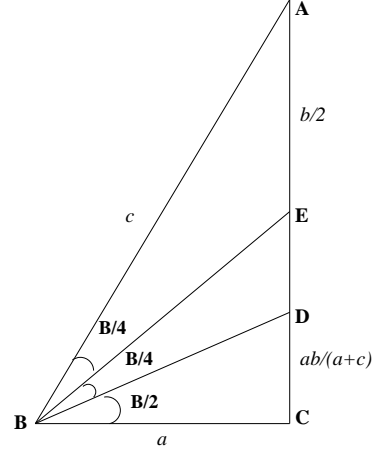
## Solution 1:

Since  $E$  is the mid-point of  $AC$ , we have  $AE = EC = b/2$ . Since  $BD$  bisects  $\angle ABC$ , we also know that  $CD = ab/(a+c)$ . Since  $BE$  bisects  $\angle ABD$ , we also have

$$\frac{BD^2}{BA^2} = \frac{DE^2}{EA^2}.$$

However,

$$\begin{aligned} BD^2 &= BC^2 + CD^2 = a^2 + \frac{a^2 b^2}{(a+c)^2}, \\ DE^2 &= \left( \frac{b}{2} - \frac{ab}{a+c} \right)^2. \end{aligned}$$



Using these in the above expression and simplifying, we get

$$a^2 \{ (a+c)^2 + b^2 \} = c^2 (c-a)^2.$$

Using  $c^2 = a^2 + b^2$  and eliminating  $b$ , we obtain

$$c^3 - 2ac^2 - a^2c - 2a^3 = 0.$$

Introducing  $t = c/a$ , this reduces to a cubic equation;

$$t^3 - 2t^2 - t - 2 = 0.$$

Consider the function  $f(t) = t^3 - 2t^2 - t - 2$  for  $t > 0$  (as  $c/a$  is positive). For  $0 < t \leq 2$ , we see that  $f(t) = t^2(t-2) - t - 2 < 0$ . We also observe that  $f(t) = (t-2)(t^2-1) - 4$  is strictly increasing on  $(2, \infty)$ . It is easy to compute

$$f(5/2) = -\frac{11}{8} < 0, \quad \text{and} \quad f(3) = 4 > 0.$$

Hence there is a unique value of  $t$  in the interval  $(5/2, 3)$  such that  $f(t) = 0$ . We conclude that

$$\frac{5}{2} < \frac{c}{a} < 3.$$

**Solution 2:** Let us take  $\angle B/4 = \theta$ . Then  $\angle EBC = \angle DBE = \theta$  and  $\angle CBD = 2\theta$ . Using sine rule in triangles  $BEA$  and  $BEC$ , we get

$$\begin{aligned} \frac{BE}{\sin A} &= \frac{AE}{\sin \theta}, \\ \frac{BE}{\sin 90^\circ} &= \frac{CE}{\sin 3\theta}. \end{aligned}$$

Since  $AE = CE$ , we obtain  $\sin 3\theta \sin A = \sin \theta$ . However  $A = 90^\circ - 4\theta$ . Thus we get  $\sin 3\theta \cos 4\theta = \sin \theta$ . Note that

$$\frac{c}{a} = \frac{1}{\cos 4\theta} = \frac{\sin 3\theta}{\sin \theta} = 3 - 4 \sin^2 \theta.$$

This shows that  $c/a < 3$ . Using  $c/a = 3 - 4 \sin^2 \theta$ , it is easy to compute  $\cos 2\theta = ((c/a) - 1)/2$ . Hence

$$\frac{a}{c} = \cos 4\theta = \frac{1}{2} \left( \frac{c}{a} - 1 \right)^2 - 1.$$

Suppose  $c/a \leq 5/2$ . Then  $((c/a) - 1)^2 \leq 9/4$  and  $a/c \geq 2/5$ . Thus

$$\frac{2}{5} \leq \frac{a}{c} = \frac{1}{2} \left( \frac{c}{a} - 1 \right)^2 - 1 \leq \frac{9}{8} - 1 = \frac{1}{8},$$

which is absurd. We conclude that  $c/a > 5/2$ .

2. Let  $n$  be a natural number such that  $n = a^2 + b^2 + c^2$ , for some natural numbers  $a, b, c$ . Prove that

$$9n = (p_1a + q_1b + r_1c)^2 + (p_2a + q_2b + r_2c)^2 + (p_3a + q_3b + r_3c)^2,$$

where  $p_j$ 's,  $q_j$ 's,  $r_j$ 's are all **nonzero** integers. Further, if 3 does **not** divide at least one of  $a, b, c$ , prove that  $9n$  can be expressed in the form  $x^2 + y^2 + z^2$ , where  $x, y, z$  are natural numbers **none** of which is divisible by 3.

**Solution:** It can be easily seen that

$$9n = (2b + 2c - a)^2 + (2c + 2a - b)^2 + (2a + 2b - c)^2.$$

Thus we can take  $p_1 = p_2 = p_3 = 2$ ,  $q_1 = q_2 = q_3 = 2$  and  $r_1 = r_2 = r_3 = -1$ . Suppose 3 does not divide  $\gcd(a, b, c)$ . Then 3 does divide at least one of  $a, b, c$ ; say 3 does not divide  $a$ . Note that each of  $2b + 2c - a$ ,  $2c + 2a - b$  and  $2a + 2b - c$  is either divisible by 3 or none of them is divisible by 3, as the difference of any two sums is always divisible by 3. If 3 does not divide  $2b + 2c - a$ , then we have the required representation. If 3 divides  $2b + 2c - a$ , then 3 does not divide  $2b + 2c + a$ . On the other hand, we also note that

$$9n = (2b + 2c + a)^2 + (2c - 2a - b)^2 + (-2a + 2b - c)^2 = x^2 + y^2 + z^2,$$

where  $x = 2b + 2c + a$ ,  $y = 2c - 2a - b$  and  $z = -2a + 2b - c$ . Since  $x - y = 3(b + a)$  and 3 does not divide  $x$ , it follows that 3 does not divide  $y$  as well. Similarly, we conclude that 3 does not divide  $z$ .

3. Let  $m$  and  $n$  be positive integers such that the equation  $x^2 - mx + n = 0$  has real roots  $\alpha$  and  $\beta$ . Prove that  $\alpha$  and  $\beta$  are integers if and only if  $[m\alpha] + [m\beta]$  is the square of an integer. (Here  $[x]$  denotes the largest integer not exceeding  $x$ .)

**Solution:** If  $\alpha$  and  $\beta$  are both integers, then

$$[m\alpha] + [m\beta] = m\alpha + m\beta = m(\alpha + \beta) = m^2.$$

This proves one implication.

Observe that  $\alpha + \beta = m$  and  $\alpha\beta = n$ . We use the property of integer function:  $x - 1 < [x] \leq x$  for any real number  $x$ . Thus

$$m^2 - 2 = m(\alpha + \beta) - 2 = m\alpha - 1 + m\beta - 1 < [m\alpha] + [m\beta] \leq m(\alpha + \beta) = m^2.$$

Since  $m$  and  $n$  are positive integers, both  $\alpha$  and  $\beta$  must be positive. If  $m \geq 2$ , we observe that there is no square between  $m^2 - 2$  and  $m^2$ . Hence, either  $m = 1$  or  $[m\alpha] + [m\beta] = m^2$ . If  $m = 1$ , then  $\alpha + \beta = 1$  implies that both  $\alpha$  and  $\beta$  are positive reals smaller than 1. Hence  $n = \alpha\beta$  cannot be a positive integer. We conclude that  $[m\alpha] + [m\beta] = m^2$ . Putting  $m = \alpha + \beta$  in this relation, we get

$$[\alpha^2 + n] + [\beta^2 + n] = (\alpha + \beta)^2.$$

Using  $[x + k] = [x] + k$  for any real number  $x$  and integer  $k$ , this reduces to

$$[\alpha^2] + [\beta^2] = \alpha^2 + \beta^2.$$

This shows that  $\alpha^2$  and  $\beta^2$  are both integers. On the other hand,

$$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = m(\alpha - \beta).$$

Thus

$$(\alpha - \beta) = \frac{\alpha^2 - \beta^2}{m},$$

is a rational number. Since  $\alpha + \beta = m$  is a rational number, it follows that both  $\alpha$  and  $\beta$  are rational numbers. However, both  $\alpha^2$  and  $\beta^2$  are integers. Hence each of  $\alpha$  and  $\beta$  is an integer.

4. Let  $\sigma = (a_1, a_2, a_3, \dots, a_n)$  be a permutation of  $(1, 2, 3, \dots, n)$ . A pair  $(a_i, a_j)$  is said to correspond to an inversion of  $\sigma$ , if  $i < j$  but  $a_i > a_j$ . (Example: In the permutation  $(2, 4, 5, 3, 1)$ , there are 6 inversions corresponding to the pairs  $(2, 1)$ ,  $(4, 3)$ ,  $(4, 1)$ ,  $(5, 3)$ ,  $(5, 1)$ ,  $(3, 1)$ .) How many permutations of  $(1, 2, 3, \dots, n)$ , ( $n \geq 3$ ), have exactly **two** inversions?

**Solution:** In a permutation of  $(1, 2, 3, \dots, n)$ , two inversions can occur in only one of the following two ways:

(A) Two disjoint consecutive pairs are interchanged:

$$\begin{aligned} (1, 2, 3, j-1, j, j+1, j+2, \dots, k-1, k, k+1, k+2, \dots, n) \\ \longrightarrow (1, 2, \dots, j-1, j+1, j, j+2, \dots, k-1, k+1, k, k+2, \dots, n). \end{aligned}$$

(B) Each block of three consecutive integers can be permuted in any of the following 2 ways;

$$\begin{aligned} (1, 2, 3, \dots, k, k+1, k+2, \dots, n) &\longrightarrow (1, 2, \dots, k+2, k, k+1, \dots, n); \\ (1, 2, 3, \dots, k, k+1, k+2, \dots, n) &\longrightarrow (1, 2, \dots, k+1, k+2, k, \dots, n). \end{aligned}$$

Consider case (A). For  $j = 1$ , there are  $n - 3$  possible values of  $k$ ; for  $j = 2$ , there are  $n - 4$  possibilities for  $k$  and so on. Thus the number of permutations with two inversions of this type is

$$1 + 2 + \dots + (n - 3) = \frac{(n - 3)(n - 2)}{2}.$$

In case (B), we see that there are  $n - 2$  permutations of each type, since  $k$  can take values from 1 to  $n - 2$ . Hence we get  $2(n - 2)$  permutations of this type.

Finally, the number of permutations with **two** inversions is

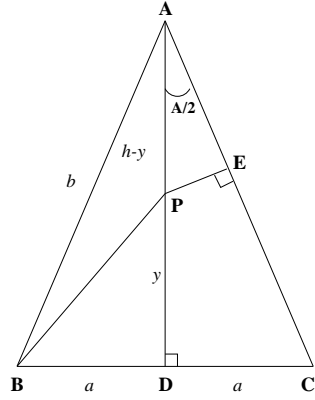
$$\frac{(n-3)(n-2)}{2} + 2(n-2) = \frac{(n+1)(n-2)}{2}.$$

5. Let  $ABC$  be a triangle in which  $AB = AC$ . Let  $D$  be the mid-point of  $BC$  and  $P$  be a point on  $AD$ . Suppose  $E$  is the foot of perpendicular from  $P$  on  $AC$ . If  $\frac{AP}{PD} = \frac{BP}{PE} = \lambda$ ,  $\frac{BD}{AD} = m$  and  $z = m^2(1 + \lambda)$ , prove that

$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0.$$

Hence show that  $\lambda \geq 2$  and  $\lambda = 2$  if and only if  $ABC$  is equilateral.

**Solution:**



Let  $AD = h$ ,  $PD = y$  and  $BD = DC = a$ . We observe that  $BP^2 = a^2 + y^2$ . Moreover,

$$PE = PA \sin \angle DAC = (h - y) \frac{DC}{AC} = \frac{a(h - y)}{b},$$

where  $b = AC = AB$ . Using  $AP/PD = (h - y)/y$ , we obtain  $y = h/(1 + \lambda)$ . Thus

$$\lambda^2 = \frac{BP^2}{PE^2} = \frac{(a^2 + y^2)b^2}{(h - y)^2 a^2}.$$

But  $(h - y) = \lambda y = \lambda h/(1 + \lambda)$  and  $b^2 = a^2 + h^2$ . Thus we obtain

$$\lambda^4 = \frac{(a^2(1 + \lambda)^2 + h^2)(a^2 + h^2)}{a^2 h^2}.$$

Using  $m = a/h$  and  $z = m^2(1 + \lambda)$ , this simplifies to

$$z^2 - z(\lambda^3 - \lambda^2 - 2) + 1 = 0.$$

Dividing by  $z$ , this gives

$$z + \frac{1}{z} = \lambda^3 - \lambda^2 - 2.$$

However  $z + (1/z) \geq 2$  for any positive real number  $z$ . Thus  $\lambda^3 - \lambda^2 - 4 \geq 0$ . This may be written in the form  $(\lambda - 2)(\lambda^2 + \lambda + 2) \geq 0$ . But  $\lambda^2 + \lambda + 2 > 0$ . (For example, one may check that its discriminant is negative.) Hence  $\lambda \geq 2$ . If  $\lambda = 2$ , then  $z + (1/z) = 2$  and hence  $z = 1$ . This gives  $m^2 = 1/3$  or  $\tan(A/2) = m = 1/\sqrt{3}$ . Thus  $A = 60^\circ$  and hence  $ABC$  is equilateral.

Conversely, if triangle  $ABC$  is equilateral, then  $m = \tan(A/2) = 1/\sqrt{3}$  and hence  $z = (1 + \lambda)/3$ . Substituting this in the equation satisfied by  $z$ , we obtain

$$(1 + \lambda)^2 - 3(1 + \lambda)(\lambda^3 - \lambda^2 - 2) + 9 = 0.$$

This may be written in the form  $(\lambda - 2)(3\lambda^3 + 6\lambda^2 + 8\lambda + 8) = 0$ . Here the second factor is positive because  $\lambda > 0$ . We conclude that  $\lambda = 2$ .

6. If  $x, y, z$  are positive real numbers, prove that

$$(x + y + z)^2(yz + zx + xy)^2 \leq 3(y^2 + yz + z^2)(z^2 + zx + x^2)(x^2 + xy + y^2).$$

**Solution 1:** We begin with the observation that

$$x^2 + xy + y^2 = \frac{3}{4}(x + y)^2 + \frac{1}{4}(x - y)^2 \geq \frac{3}{4}(x + y)^2,$$

and similar bounds for  $y^2 + yz + z^2$ ,  $z^2 + zx + x^2$ . Thus

$$3(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) \geq \frac{81}{64}(x + y)^2(y + z)^2(z + x)^2.$$

Thus it is sufficient to prove that

$$(x + y + z)(xy + yz + zx) \leq \frac{9}{8}(x + y)(y + z)(z + x).$$

Equivalently, we need to prove that

$$8(x + y + z)(xy + yz + zx) \leq 9(x + y)(y + z)(z + x).$$

However, we note that

$$(x + y)(y + z)(z + x) = (x + y + z)(yz + zx + xy) - xyz.$$

Thus the required inequality takes the form

$$(x + y)(y + z)(z + x) \geq 8xyz.$$

This follows from AM-GM inequalities;

$$x + y \geq 2\sqrt{xy}, \quad y + z \geq 2\sqrt{yz}, \quad z + x \geq 2\sqrt{zx}.$$

**Solution 2:** Let us introduce  $x + y = c$ ,  $y + z = a$  and  $z + x = b$ . Then  $a, b, c$  are the sides of a triangle. If  $s = (a + b + c)/2$ , then it is easy to calculate  $x = s - a$ ,  $y = s - b$ ,  $z = s - c$  and  $x + y + z = s$ . We also observe that

$$x^2 + xy + y^2 = (x + y)^2 - xy = c^2 - \frac{1}{4}(c + a - b)(c + b - a) = \frac{3}{4}c^2 + \frac{1}{4}(a - b)^2 \geq \frac{3}{4}c^2.$$

Moreover,  $xy + yz + zx = (s - a)(s - b) + (s - b)(s - c) + (s - c)(s - a)$ . Thus it is sufficient to prove that

$$s \sum (s - a)(s - b) \leq \frac{9}{8}abc.$$

But,  $\sum (s - a)(s - b) = r(4R + r)$ , where  $r, R$  are respectively the in-radius, the circum-radius of the triangle whose sides are  $a, b, c$ , and  $abc = 4Rrs$ . Thus the inequality reduces to

$$r(4R + r) \leq \frac{9}{2}Rr.$$

This is simply  $2r \leq R$ . This follows from  $IO^2 = R(R - 2r)$ , where  $I$  is the incentre and  $O$  the circumcentre.

**Solution 3:** If we set  $x = \lambda a$ ,  $y = \lambda b$ ,  $z = \lambda c$ , then the inequality changes to

$$(a + b + c)^2(ab + bc + ca)^2 \leq 3(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2).$$

This shows that we may assume  $x + y + z = 1$ . Let  $\alpha = xy + yz + zx$ . We see that

$$\begin{aligned} x^2 + xy + y^2 &= (x + y)^2 - xy \\ &= (x + y)(1 - z) - xy \\ &= x + y - \alpha = 1 - z - \alpha. \end{aligned}$$

Thus

$$\begin{aligned} \prod(x^2 + xy + y^2) &= (1 - \alpha - z)(1 - \alpha - x)(1 - \alpha - y) \\ &= (1 - \alpha)^3 - (1 - \alpha)^2 + (1 - \alpha)\alpha - xyz \\ &= \alpha^2 - \alpha^3 - xyz. \end{aligned}$$

Thus we need to prove that  $\alpha^2 \leq 3(\alpha^2 - \alpha^3 - xyz)$ . This reduces to

$$3xyz \leq \alpha^2(2 - 3\alpha).$$

However

$$3\alpha = 3(xy + yz + zx) \leq (x + y + z)^2 = 1,$$

so that  $2 - 3\alpha \geq 1$ . Thus it suffices to prove that  $3xyz \leq \alpha^2$ . But

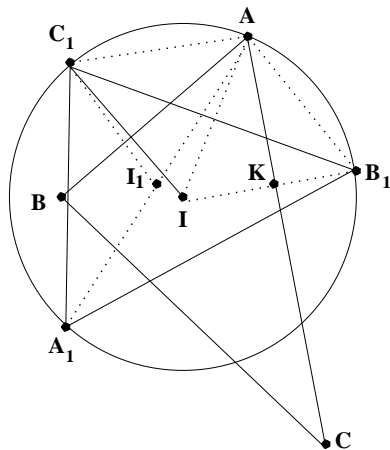
$$\begin{aligned} \alpha^2 - 3xyz &= (xy + yz + zx)^2 - 3xyz(x + y + z) \\ &= \sum_{\text{cyclic}} x^2y^2 - xyz(x + y + z) \\ &= \frac{1}{2} \sum_{\text{cyclic}} (xy - yz)^2 \geq 0. \end{aligned}$$


---

## Problems and Solutions of INMO-2008

1. Let  $ABC$  be a triangle,  $I$  its in-centre;  $A_1, B_1, C_1$  be the reflections of  $I$  in  $BC, CA, AB$  respectively. Suppose the circum-circle of triangle  $A_1B_1C_1$  passes through  $A$ . Prove that  $B_1, C_1, I, I_1$  are concyclic, where  $I_1$  is the in-centre of triangle  $A_1B_1C_1$ .

**Solution:**



Note that  $IA_1 = IB_1 = IC_1 = 2r$ , where  $r$  is the in-radius of the triangle  $ABC$ . Hence  $I$  is the circum-centre of the triangle  $A_1B_1C_1$ .

Let  $K$  be the point of intersection of  $IB_1$  and  $AC$ . Then  $IK = r$ ,  $IA = 2r$  and  $\angle IKA = 90^\circ$ . It follows that  $\angle IAK = 30^\circ$  and hence  $\angle IAB_1 = 60^\circ$ . Thus  $AIB_1$  is an equilateral triangle. Similarly triangle  $AIC_1$  is also equilateral. We hence obtain  $AB_1 = AC_1 = AI = IB_1 = IC_1 = 2r$ .

We also observe that  $\angle B_1IC_1 = 120^\circ$  and  $IB_1AC_1$  is a rhombus. Thus  $\angle B_1AC_1 = 120^\circ$  and by concyclicity  $\angle A_1 = 60^\circ$ . Since  $AB_1 = AC_1$ ,  $A$  is the midpoint of the arc  $B_1AC_1$ . It follows that  $A_1A$  bisects  $\angle A_1$  and  $I_1$  lies on the line  $A_1A$ . This implies that

$$\angle B_1I_1C_1 = 90^\circ + \angle A_1/2 = 90^\circ + 30^\circ = 120^\circ.$$

Since  $\angle B_1IC_1 = 120^\circ$ , we conclude that  $B_1, I, I_1, C_1$  are concyclic. (Further  $A$  is the centre.)

2. Find all triples  $(p, x, y)$  such that  $p^x = y^4 + 4$ , where  $p$  is a prime and  $x, y$  are natural numbers.

**Solution:** We begin with the standard factorisation

$$y^4 + 4 = (y^2 - 2y + 2)(y^2 + 2y + 2).$$

Thus we have  $y^2 - 2y + 2 = p^m$  and  $y^2 + 2y + 2 = p^n$  for some positive integers  $m$  and  $n$  such that  $m + n = x$ . Since  $y^2 - 2y + 2 < y^2 + 2y + 2$ , we have  $m < n$  so that  $p^m$  divides  $p^n$ . Thus  $y^2 - 2y + 2$  divides  $y^2 + 2y + 2$ . Writing  $y^2 + 2y + 2 = y^2 - 2y + 2 + 4y$ , we infer that  $y^2 - 2y + 2$  divides  $4y$  and hence  $y^2 - 2y + 2$  divides  $4y^2$ . But

$$4y^2 = 4(y^2 - 2y + 2) + 8(y - 1).$$

Thus  $y^2 - 2y + 2$  divides  $8(y - 1)$ . Since  $y^2 - 2y + 2$  divides both  $4y$  and  $8(y - 1)$ , we conclude that it also divides 8. This gives  $y^2 - 2y + 2 = 1, 2, 4$  or  $8$ .

If  $y^2 - 2y + 2 = 1$ , then  $y = 1$  and  $y^4 + 4 = 5$ , giving  $p = 5$  and  $x = 1$ . If  $y^2 - 2y + 2 = 2$ , then  $y^2 - 2y = 0$  giving  $y = 2$ . But then  $y^4 + 4 = 20$  is not the power of a prime. The equations  $y^2 - 2y + 2 = 4$  and  $y^2 - 2y + 2 = 8$  have no integer solutions. We conclude that  $(p, x, y) = (5, 1, 1)$  is the only solution.

Alternatively, using  $y^2 - 2y + 2 = p^m$  and  $y^2 + 2y + 2 = p^n$ , we may get

$$4y = p^m(p^{n-m} - 1).$$

If  $m > 0$ , then  $p$  divides 4 or  $y$ . If  $p$  divides 4, then  $p = 2$ . If  $p$  divides  $y$ , then  $y^2 - 2y + 2 = p^m$  shows that  $p$  divides 2 and hence  $p = 2$ . But then  $2^x = y^4 + 4$ , which shows that  $y$  is even. Taking  $y = 2z$ , we get  $2^{x-2} = 4z^4 + 1$ . This implies that  $z = 0$  and hence  $y = 0$ , which is a contradiction. Thus  $m = 0$  and  $y^2 - 2y + 2 = 1$ . This gives  $y = 1$  and hence  $p = 5, x = 1$ .

3. Let  $A$  be a set of real numbers such that  $A$  has at least four elements. Suppose  $A$  has the property that  $a^2 + bc$  is a rational number for all distinct numbers  $a, b, c$  in  $A$ . Prove that there exists a positive integer  $M$  such that  $a\sqrt{M}$  is a rational number for every  $a$  in  $A$ .

**Solution:** Suppose  $0 \in A$ . Then  $a^2 = a^2 + 0 \times b$  is rational and  $ab = 0^2 + ab$  is also rational for all  $a, b$  in  $A$ ,  $a \neq 0$ ,  $b \neq 0$ ,  $a \neq b$ . Hence  $a = a_1\sqrt{M}$  for some rational  $a_1$  and natural number  $M$ . For any  $b \neq 0$ , we have

$$b\sqrt{M} = \frac{ab}{a_1}.$$

which is a rational number.

Hence we may assume  $0$  is not in  $A$ . If there is a number  $a$  in  $A$  such that  $-a$  is also in  $A$ , then again we can get the conclusion as follows. Consider two other elements  $c, d$  in  $A$ . Then  $c^2 + da$  is rational and  $c^2 - da$  is also rational. It follows that  $c^2$  is rational and  $da$  is rational. Similarly,  $d^2$  and  $ca$  are also rationals. Thus  $d/c = (da)/(ca)$  is rational. Note that we can vary  $d$  over  $A$  with  $d \neq c$  and  $d \neq a$ . Again  $c^2$  is rational implies that  $c = c_1\sqrt{M}$  for some rational  $c_1$  and natural number  $M$ . We observe that  $c\sqrt{M} = c_1M$  is rational, and

$$a\sqrt{M} = \frac{ca}{c_1},$$

so that  $a\sqrt{M}$  is a rational number. Similarly is the case with  $-a\sqrt{M}$ . For any other element  $d$ ,

$$b\sqrt{M} = Mc_1 \frac{d}{c}$$

is a rational number.

Thus we may now assume that  $0$  is not in  $A$  and  $a + b \neq 0$  for any  $a, b$  in  $A$ . Let  $a, b, c, d$  be four distinct elements of  $A$ . We may assume  $|a| > |b|$ . Then  $d^2 + ab$  and  $d^2 + bc$  are rational numbers and so is their difference  $ab - bc$ . Writing  $a^2 + ab = a^2 + bc + (ab - bc)$ , and using the facts  $a^2 + bc$ ,  $ab - bc$  are rationals, we conclude that  $a^2 + ab$  is also a rational number. Similarly,  $b^2 + ab$  is also a rational number.

Consider

$$q = \frac{a}{b} = \frac{a^2 + ab}{b^2 + ab}.$$

Note that  $a^2 + ab > 0$ . Thus  $q$  is a rational number and  $a = bq$ . This gives  $a^2 + ab = b^2(q^2 + q)$ . Let us take  $b^2(q^2 + q) = l$ . Then

$$|b| = \sqrt{\frac{l}{q^2 + q}} = \sqrt{\frac{x}{y}},$$

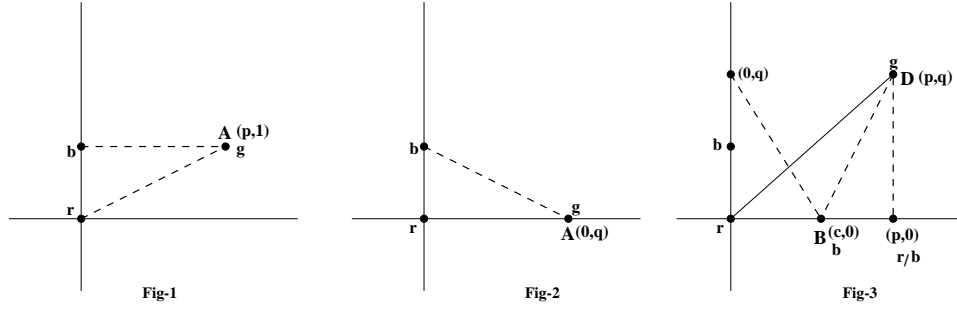
where  $x$  and  $y$  are natural numbers. Take  $M = xy$ . Then  $|b|\sqrt{M} = x$  is a rational number. Finally, for any  $c$  in  $A$ , we have

$$c\sqrt{M} = b\sqrt{M} \frac{c}{b},$$

is also a rational number.

4. All the points with integer coordinates in the  $xy$ -plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point  $(0, 0)$  is coloured red and the point  $(0, 1)$  is coloured blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a **right-angled** triangle.

**Solution:** Consider the lattice points (points with integer coordinates) on the lines  $y = 0$  and  $y = 1$ , other than  $(0, 0)$  and  $(0, 1)$ . If one of them, say  $A = (p, 1)$ , is coloured green, then we have a right-angled triangle with  $(0, 0)$ ,  $(0, 1)$  and  $A$  as vertices, all having different colours. (See Figures 1 and 2.)



If not, the lattice points on  $y = 0$  and  $y = 1$  are all red or blue. We consider three different cases.

**Case 1.** Suppose a point  $B = (c, 0)$  is blue. Consider a green point  $D = (p, q)$  in the plane. Suppose  $p \neq 0$ . If its projection  $(p, 0)$  on the  $x$ -axis is red, then  $(p, q)$ ,  $(p, 0)$  and  $(c, 0)$  are the vertices of a required type of right-angled triangle. If  $(p, 0)$  is blue, then we can consider the triangle whose vertices are  $(0, 0)$ ,  $(p, 0)$  and  $(p, q)$ . If  $p = 0$ , then the points  $D$ ,  $(0, 0)$  and  $(c, 0)$  will work. (Figure 3.)

**Case 2.** A point  $D = (c, 1)$ , on the line  $y = 1$ , is red. A similar argument works in this case.

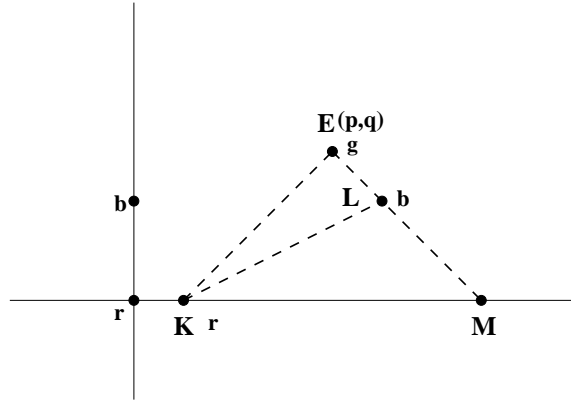
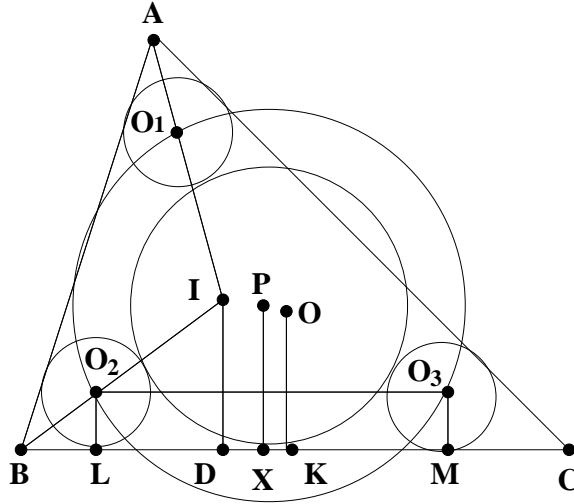


Fig-4

**Case 3.** Suppose all the lattice points on the line  $y = 0$  are red and all on the line  $y = 1$  are blue points. Consider a green point  $E = (p, q)$ , where  $q \neq 0$  and  $q \neq 1$ . (See Figure 4.) Consider an isosceles right-angled triangle  $EKM$  with  $\angle E = 90^\circ$  such that the hypotenuse  $KM$  is a part of the  $x$ -axis. Let  $EM$  intersect  $y = 1$  in  $L$ . Then  $K$  is a red point and  $L$  is a blue point. Hence  $EKL$  is a desired triangle.

5. Let  $ABC$  be a triangle;  $\Gamma_A$ ,  $\Gamma_B$ ,  $\Gamma_C$  be three equal, disjoint circles inside  $ABC$  such that  $\Gamma_A$  touches  $AB$  and  $AC$ ;  $\Gamma_B$  touches  $AB$ ; and  $BC$ , and  $\Gamma_C$  touches  $BC$  and  $CA$ . Let  $\Gamma$  be a circle touching circles  $\Gamma_A$ ,  $\Gamma_B$ ,  $\Gamma_C$  externally. Prove that the line joining the circum-centre  $O$  and the in-centre  $I$  of triangle  $ABC$  passes through the centre of  $\Gamma$ .

**Solution:** Let  $O_1$ ,  $O_2$ ,  $O_3$  be the centres of the circles  $\Gamma_A$ ,  $\Gamma_B$ ,  $\Gamma_C$  respectively, and let  $P$  be the circum-centre of the triangle  $O_1O_2O_3$ . Let  $x$  denote the common radius of three circles  $\Gamma_A$ ,  $\Gamma_B$ ,  $\Gamma_C$ . Note that  $P$  is also the centre of the circle  $\Gamma$ , as  $O_1P$ ,  $O_2P$ ,  $O_3P$  each exceed the radius of  $\Gamma$  by  $x$ . Let  $D$ ,  $X$ ,  $K$ ,  $L$ ,  $M$  be respectively the projections of  $I$ ,  $P$ ,  $O$ ,  $O_1$ ,  $O_2$  on  $BC$ .



From  $\frac{BL}{BD} = \frac{LO_2}{DI}$ , we get  $BL = x(s-b)/r$ , as  $ID = r$  and  $BD = (s-b)$ . Similarly,  $CM = x(s-c)/r$ . Therefore,  $LM = a - \frac{x}{r}(s-b + s-c) = \frac{a}{r}(r-x)$ . Since  $O_2LMO_3$  is a rectangle and  $PX$  is the perpendicular bisector of  $O_2O_3$ , it is perpendicular bisector of  $LM$  as well. Thus

$$\begin{aligned} LX &= \frac{1}{2}LM = \frac{a}{2r}(r-x); \\ BX &= BL + LX = \frac{x}{r}(s-b) + \frac{a}{2r}(r-x) = \frac{a}{2} - \frac{x(b-c)}{2r}; \\ DK &= BK - BD = \frac{a}{2} - (s-b) = \frac{b-c}{2}; \\ XK &= BK - BX = \frac{a}{2} - \frac{a}{2} + \frac{x(b-c)}{2r} = \frac{x(b-c)}{2r}. \end{aligned}$$

Hence we get

$$\frac{XK}{DK} = \frac{x}{r}.$$

We observe that the sides of triangle  $O_1O_2O_3$  are

$$O_2O_3 = LM = \frac{a}{r}(r-x), \quad O_3O_1 = \frac{b}{r}(r-x), \quad O_1O_2 = \frac{c}{r}(r-x).$$

Thus the sides of  $O_1O_2O_3$  and those of  $ABC$  are in the ratio  $(r-x)/r$ . Further, as the sides of  $O_1O_2O_3$  are parallel to those of  $ABC$ , we see that  $I$  is the in-centre of  $O_1O_2O_3$  as well. This gives  $IP/IO = (r-x)/r$ , and hence  $PO/IO = x/r$ . Thus we obtain

$$\frac{XK}{DK} = \frac{PO}{IO}.$$

It follows that  $I, P, O$  are collinear.

Alternately, we also infer that  $I$  is the centre of homothety which takes the figure  $O_1O_2O_3$  to  $ABC$ . Hence it takes  $P$  to  $O$ . It follows that  $I, P, O$  are collinear

6. Let  $P(x)$  be a given polynomial with integer coefficients. Prove that there exist two polynomials  $Q(x)$  and  $R(x)$ , again with integer coefficients, such that (i)  $P(x)Q(x)$  is a polynomial in  $x^2$ ; and (ii)  $P(x)R(x)$  is a polynomial in  $x^3$ .

**Solution:** Let  $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$  be a polynomial with integer coefficients.

**Part (i)** We may write

$$P(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + x(a_1 + a_3x^2 + a_5x^4 + \cdots).$$

Define

$$Q(x) = a_0 + a_2x^2 + a_4x^4 + \cdots - x(a_1 + a_3x^2 + a_5x^4 + \cdots).$$

Then  $Q(x)$  is also a polynomial with integer coefficients and

$$P(x)Q(x) = (a_0 + a_2x^2 + a_4x^4 + \cdots)^2 - x^2(a_1 + a_3x^2 + a_5x^4 + \cdots)^2$$

is a polynomial in  $x^2$ .

**Part (ii)** We write again

$$P(x) = A(x) + xB(x) + x^2C(x),$$

where

$$\begin{aligned} A(x) &= a_0 + a_3x^3 + a_6x^6 + \cdots, \\ B(x) &= a_1 + a_4x^3 + a_7x^6 + \cdots, \\ C(x) &= a_2 + a_5x^3 + a_8x^6 + \cdots. \end{aligned}$$

Note that  $A(x)$ ,  $B(x)$  and  $C(x)$  are polynomials with integer coefficients and each of these is a polynomial in  $x^3$ . We may introduce

$$\begin{aligned} S(x) &= A(x) + \omega xB(x) + \omega^2 x^2C(x), \\ T(x) &= A(x) + \omega^2 xB(x) + \omega x^2C(x), \end{aligned}$$

where  $\omega$  is an imaginary cube-root of unity. Then

$$\begin{aligned} S(x)T(x) &= (A(x))^2 + x^2(B(x))^2 + x^4(C(x))^2 \\ &\quad - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x) \end{aligned}$$

since  $\omega^3 = 1$  and  $\omega + \omega^2 = -1$ . Taking  $R(x) = S(x)T(x)$ , we obtain

$$P(x)R(x) = (A(x))^3 + x^3(B(x))^3 + x^6(C(x))^3 - 3x^3A(x)B(x)C(x),$$

which is a polynomial in  $x^3$ . This follows from the identity

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc.$$

Alternately,  $R(x)$  may be directly defined by

$$\begin{aligned} R(x) &= (A(x))^2 + x^2(B(x))^2 + x^4(C(x))^2 \\ &\quad - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x). \end{aligned}$$

# 24th Indian National Mathematical Olympiad, 2009

## Problems and Solutions

- Let  $ABC$  be a triangle and let  $P$  be an interior point such that  $\angle BPC = 90^\circ$ ,  $\angle BAP = \angle BCP$ . Let  $M, N$  be the mid-points of  $AC, BC$  respectively. Suppose  $BP = 2PM$ . Prove that  $A, P, N$  are collinear.

### Solution:

Extend  $CP$  to  $D$  such that  $CP = PD$ . Let  $\angle BCP = \alpha = \angle BAP$ . Observe that  $BP$  is the perpendicular bisector of  $CD$ . Hence  $BC = BD$  and  $BCD$  is an isosceles triangle. Thus  $\angle BDP = \alpha$ . But then  $\angle BDP = \alpha = \angle BAP$ . This implies that  $B, P, A, D$  all lie on a circle. In turn, we conclude that  $\angle DAB = \angle DPB = 90^\circ$ . Since  $P$  is the mid-point of  $CD$  (by construction) and  $M$  is the mid-point of  $CA$  (given), it follows that  $PM$  is parallel to  $DA$  and  $DA = 2PM = BP$ . Thus  $DBPA$  is an isosceles trapezium and  $DB$  is parallel to  $PA$ .

We hence get

$$\angle DPA = \angle BAP = \angle BCP = \angle NPC;$$

the last equality follows from the fact that  $\angle BPC = 90^\circ$ , and  $N$  is the mid-point of  $CB$  so that  $NP = NC = NB$  for the right-angled triangle  $BPC$ . It follows that  $A, P, N$  are collinear.

### Alternate Solution:

We use coordinate geometry. Let us take  $P = (0, 0)$ , and the coordinate axes along  $PC$  and  $PB$ ; We take  $C = (c, 0)$  and  $B = (0, b)$ . Let  $A = (u, v)$ . We see that  $N = (c/2, b/2)$  and  $M = ((u+c)/2, v/2)$ . The condition  $PB = 2PM$  translates to

$$(u+c)^2 + v^2 = b^2.$$

We observe that the slope of  $CP = 0$ ; that of  $CB$  is  $-b/c$ ; that of  $PA$  is  $v/u$ ; and that of  $BA$  is  $(v-b)/u$ . Taking proper signs, we can convert  $\angle PCB = \angle PAB$ , via  $\tan$  function, to the following relation:

$$u^2 + v^2 - vb = -cu.$$

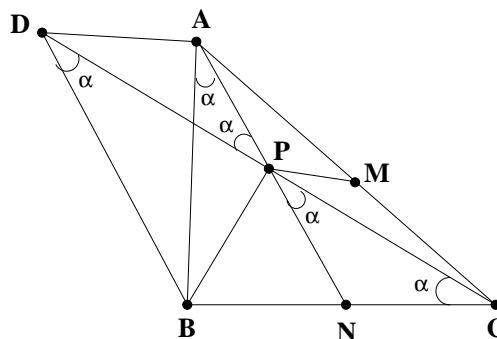
Thus we obtain

$$u(u+c) = v(b-v), \quad c(c+u) = b(b-v).$$

It follows that  $v/u = b/c$ . But then we get that the slope of  $AP$  and  $PN$  are the same. We conclude that  $A, P, N$  are collinear.

- Define a sequence  $\langle a_n \rangle_{n=1}^\infty$  as follows:

$$a_n = \begin{cases} 0, & \text{if the number of positive divisors of } n \text{ is odd,} \\ 1, & \text{if the number of positive divisors of } n \text{ is even.} \end{cases}$$



(The positive divisors of  $n$  include 1 as well as  $n$ .) Let  $x = 0.a_1a_2a_3\ldots$  be the real number whose decimal expansion contains  $a_n$  in the  $n$ -th place,  $n \geq 1$ . Determine, with proof, whether  $x$  is rational or irrational.

**Solution:**

We show that  $x$  is irrational. Suppose that  $x$  is rational. Then the sequence  $\langle a_n \rangle_{n=1}^\infty$  is periodic after some stage; there exist natural numbers  $k, l$  such that  $a_n = a_{n+l}$  for all  $n \geq k$ . Choose  $m$  such that  $ml \geq k$  and  $ml$  is a perfect square. Let

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad l = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r},$$

be the prime decompositions of  $m, l$  so that  $\alpha_j + \beta_j$  is even for  $1 \leq j \leq r$ . Now take a prime  $p$  different from  $p_1, p_2, \ldots, p_r$ . Consider  $ml$  and  $pml$ . Since  $pml - ml$  is divisible by  $l$ , we have  $a_{pml} = a_{ml}$ . Hence  $d(pml)$  and  $d(ml)$  have same parity. But  $d(pml) = 2d(ml)$ , since  $\gcd(p, ml) = 1$  and  $p$  is a prime. Since  $ml$  is a square,  $d(ml)$  is odd. It follows that  $d(pml)$  is even and hence  $a_{pml} \neq a_{ml}$ . This contradiction implies that  $x$  is irrational.

**Alternative Solution:** As earlier, assume that  $x$  is rational and choose natural numbers  $k, l$  such that  $a_n = a_{n+l}$  for all  $n \geq k$ . Consider the numbers  $a_{m+1}, a_{m+2}, \ldots, a_{m+l}$ , where  $m \geq k$  is any number. This must contain at least one 0. Otherwise  $a_n = 1$  for all  $n \geq k$ . But  $a_r = 0$  if and only if  $r$  is a square. Hence it follows that there are no squares for  $n > k$ , which is absurd. Thus every  $l$  consecutive terms of the sequence  $\langle a_n \rangle$  must contain a 0 after certain stage. Let  $t = \max\{k, l\}$ , and consider  $t^2$  and  $(t+1)^2$ . Since there are no squares between  $t^2$  and  $(t+1)^2$ , we conclude that  $a_{t^2+j} = 1$  for  $1 \leq j \leq 2t$ . But then, we have  $2t(> l)$  consecutive terms of the sequence  $\langle a_n \rangle$  which miss 0, contradicting our earlier observation.

3. Find all real numbers  $x$  such that

$$[x^2 + 2x] = [x]^2 + 2[x].$$

(Here  $[x]$  denotes the largest integer not exceeding  $x$ .)

**Solution:**

Adding 1 both sides, the equation reduces to

$$[(x+1)^2] = ([x+1])^2;$$

we have used  $[x] + m = [x + m]$  for every integer  $m$ . Suppose  $x + 1 \leq 0$ . Then  $[x + 1] \leq x + 1 \leq 0$ . Thus

$$([x+1])^2 \geq (x+1)^2 \geq [(x+1)^2] = ([x+1])^2.$$

Thus equality holds everywhere. This gives  $[x + 1] = x + 1$  and thus  $x + 1$  is an integer. Using  $x + 1 \leq 0$ , we conclude that

$$x \in \{-1, -2, -3, \ldots\}.$$

Suppose  $x + 1 > 0$ . We have

$$(x+1)^2 \geq [(x+1)^2] = ([x+1])^2.$$

Moreover, we also have

$$(x+1)^2 \leq 1 + [(x+1)^2] = 1 + ([x+1])^2.$$

Thus we obtain

$$[x] + 1 = [x + 1] \leq (x + 1) < \sqrt{1 + ([x + 1])^2} = \sqrt{1 + ([x] + 1)^2}.$$

This shows that

$$x \in [n, \sqrt{1 + (n + 1)^2} - 1),$$

where  $n \geq -1$  is an integer. Thus the solution set is

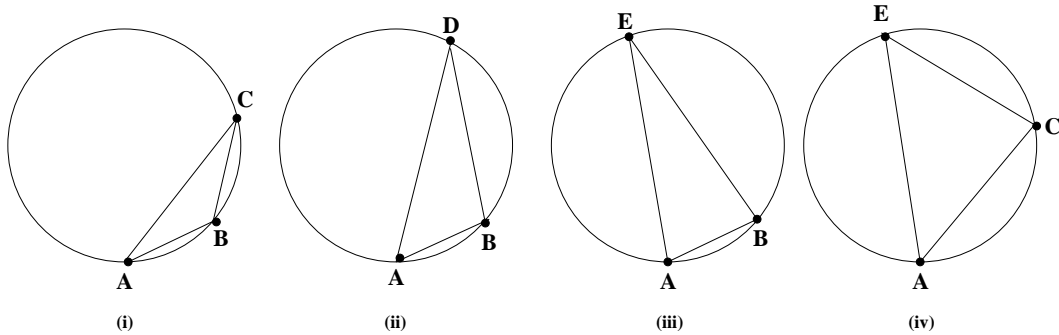
$$\{-1, -2, -3, \dots\} \cup \left\{ \bigcup_{n=-1}^{\infty} [n, \sqrt{1 + (n + 1)^2} - 1) \right\}.$$

It is easy to verify that all the real numbers in this set indeed satisfy the given equation.

4. All the points in the plane are coloured using three colours. Prove that there exists a triangle with vertices having the same colour such that *either* it is isosceles *or* its angles are in geometric progression.

**Solution:**

Consider a circle of positive radius in the plane and inscribe a regular heptagon  $ABCDEFG$  in it. Since the seven vertices of this heptagon are coloured by three colours, some three vertices have the same colour, by pigeon-hole principle. Consider the triangle formed by these three vertices. Let us call the part of the circumference separated by any two consecutive vertices of the heptagon an *arc*. The three vertices of the same colour are separated by arcs of length  $l, m, n$  as we move, say counter-clockwise, along the circle, starting from a fixed vertex among these three, where  $l + m + n = 7$ . Since, the order of  $l, m, n$  does not matter for a triangle, there are four possibilities:  $1+1+5=7$ ;  $1+2+4=7$ ;  $1+3+3=7$ ;  $2+2+3=7$ . In the first, third and fourth cases, we have isosceles triangles. In the second case, we have a triangle whose angles are in geometric progression. The four corresponding figures are shown below.

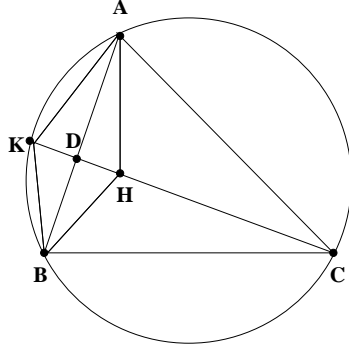


In (i),  $AB = BC$ ; in (iii),  $AE = BE$ ; in (iv),  $AC = CE$ ; and in (ii) we see that  $\angle D = \pi/7$ ,  $\angle A = 2\pi/7$  and  $\angle B = 4\pi/7$  which are in geometric progression.

5. Let  $ABC$  be an acute-angled triangle and let  $H$  be its ortho-centre. Let  $h_{\max}$  denote the largest altitude of the triangle  $ABC$ . Prove that

$$AH + BH + CH \leq 2h_{\max}.$$

**Solution:**



Let  $\angle C$  be the smallest angle, so that  $CA \geq AB$  and  $CB \geq AB$ . In this case the altitude through  $C$  is the longest one. Let the altitude through  $C$  meet  $AB$  in  $D$  and let  $H$  be the ortho-centre of  $ABC$ . Let  $CD$  extended meet the circum-circle of  $ABC$  in  $K$ . We have  $CD = h_{\max}$  so that the inequality to be proved is

$$AH + BH + CH \leq 2CD.$$

Using  $CD = CH + HD$ , this reduces to  $AH + BH \leq CD + HD$ . However, we observe that  $AH = AK$ ,  $BH = BK$  and  $HD = DK$ . (For example  $BH = BK$  and  $DH = DK$  follow from the congruency of the right-angled triangles  $DBK$  and  $DBH$ .)

Thus we need to prove that  $AK + BK \leq CK$ . Applying Ptolemy's theorem to the cyclic quadrilateral  $BCAK$ , we get

$$AB \cdot CK = AC \cdot BK + BC \cdot AK \geq AB \cdot BK + AB \cdot AK.$$

This implies that  $CK \geq AK + BK$ , which is precisely what we are looking for.

There were other beautiful solutions given by students who participated in INMO-2009. We record them here.

1. Let  $AD$ ,  $BE$ ,  $CF$  be the altitudes and  $H$  be the ortho-centre. Observe that

$$\frac{AH}{AD} = \frac{[AHB]}{[ADB]} = \frac{[AHC]}{[ADC]}.$$

This gives

$$\frac{AH}{AD} = \frac{[AHB] + [AHC]}{[ADB] + [ADC]} = 1 - \frac{[BHC]}{[ABC]}.$$

Similar expressions for the ratios  $BH/BE$  and  $CH/CF$  may be obtained. Adding, we get

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

Suppose  $AD$  is the largest altitude. We get

$$\frac{AH}{AD} + \frac{BH}{AD} + \frac{CH}{AD} \leq \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

This gives the result.

2. Let  $O$  be the circum-centre and let  $L$ ,  $M$ ,  $N$  be the mid-points of  $BC$ ,  $CA$ ,  $AB$  respectively. Then we know that  $AH = 2OL$ ,  $BH = 2OM$  and  $CH = 2ON$ . As earlier, assume  $AD$  is the largest altitude. Then  $BC$  is the least side. We have

$$\begin{aligned} 4[ABC] &= 4[BOC] + 4[COA] + 4[AOB] = BC \times 2OL + CA \times 2OM + AB \times 2ON \\ &= BC \times AH + CA \times BH + AB \times CH \\ &\geq AB(AH + BH + CH). \end{aligned}$$

Thus

$$AH + BH + CH \leq \frac{4[ABC]}{AB} = 2AD.$$

**3.** We make use of the fact that  $AH = 2R \cos \angle A$ ,  $BH = 2R \cos \angle B$ ,  $CH = 2R \cos \angle C$  and  $AD = 2R \sin \angle B \sin \angle C$ , where  $R$  is the circum-radius of  $ABC$ . We are assuming that  $AD$  is the largest altitude so that  $\angle A$  is the least angle. Thus we have to prove that

$$\cos \angle A + \cos \angle B + \cos \angle C \leq 2 \sin \angle B \angle C,$$

under the assumption  $\angle A \leq \angle B$  and  $\angle A \leq \angle C$ . On multiplying this by  $2 \sin \angle A$ , this is equivalent to

$$\begin{aligned} 2(\sin \angle A \cos \angle A + \sin \angle A \cos \angle B + \sin \angle A \cos \angle C) \\ \leq 4 \sin \angle A \sin \angle B \angle C = \sin 2A + \sin 2B + \sin 2C. \end{aligned}$$

This is equivalent to

$$\cos \angle B(\sin \angle A - \sin \angle B) + \cos \angle C(\sin \angle A - \sin \angle C) \leq 0.$$

Since  $ABC$  is acute-angled and  $A$  is the least angle, the result follows.

6. Let  $a, b, c$  be positive real numbers such that  $a^3 + b^3 = c^3$ . Prove that

$$a^2 + b^2 - c^2 > 6(c - a)(c - b).$$

**Solution:**

The given inequality may be written in the form

$$7c^2 - 6(a + b)c - (a^2 + b^2 - 6ab) < 0.$$

Putting  $x = 7c^2$ ,  $y = -6(a + b)c$ ,  $z = -(a^2 + b^2 - 6ab)$ , we have to prove that  $x + y + z < 0$ . Observe that  $x, y, z$  are not all equal ( $x > 0$ ,  $y < 0$ ). Using the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2],$$

we infer that it is sufficient to prove  $x^3 + y^3 + z^3 - 3xyz < 0$ . Substituting the values of  $x, y, z$ , we see that this is equivalent to

$$343c^6 - 216(a + b)^3c^3 - (a^2 + b^2 - 6ab)^3 - 126c^3(a + b)(a^2 + b^2 - 6ab) < 0.$$

Using  $c^3 = a^3 + b^3$ , this reduces to

$$343(a^3 + b^3)^2 - 216(a + b)^3(a^3 + b^3) - (a^2 + b^2 - 6ab)^3 - 126((a^3 + b^3)(a + b)(a^2 + b^2 - 6ab)) < 0.$$

This may be simplified (after some tedious calculations) to,

$$-a^2b^2(129a^2 - 254ab + 129b^2) < 0.$$

But  $129a^2 - 254ab + 129b^2 = 129(a - b)^2 + 4ab > 0$ . Hence the result follows.

**Remark:** The best constant  $\theta$  in the inequality  $a^2 + b^2 - c^2 \geq \theta(c - a)(c - b)$ , where  $a, b, c$

are positive reals such that  $a^3 + b^3 = c^3$ , is  $\theta = 2(1 + 2^{1/3} + 2^{-1/3})$ .

Here again, there were some beautiful solutions given by students.

1. We have

$$a^3 = c^3 - b^3 = (c - b)(c^2 + cb + b^2),$$

which is same as

$$\frac{a^2}{c - b} = \frac{c^2 + cb + b^2}{a}.$$

Similarly, we get

$$\frac{b^2}{c - a} = \frac{c^2 + ca + a^2}{b}.$$

We observe that

$$\frac{a^2}{c - b} + \frac{b^2}{c - a} = \frac{c(a^2 + b^2) - a^3 - b^3}{(c - a)(c - b)} = \frac{c(a^2 + b^2 - c^2)}{(c - a)(c - b)}.$$

This shows that

$$\frac{a^2 + b^2 - c^2}{(c - a)(c - b)} = \frac{c^2 + cb + b^2}{ca} + \frac{c^2 + ca + a^2}{cb}.$$

Thus it is sufficient to prove that

$$\frac{c^2 + cb + b^2}{ca} + \frac{c^2 + ca + a^2}{cb} \geq 6.$$

However, we have  $c^2 + b^2 \geq 2cb$  and  $c^2 + a^2 \geq 2ca$ . Hence

$$\frac{c^2 + cb + b^2}{ca} + \frac{c^2 + ca + a^2}{cb} \geq 3 \left( \frac{b}{a} + \frac{a}{b} \right) \geq 3 \times 2 = 6.$$

We have used AM-GM inequality.

2. Let us set  $x = a/c$  and  $y = b/c$ . Then  $x^3 + y^3 = 1$  and the inequality to be proved is  $x^2 + y^2 - 1 > 6(1 - x)(1 - y)$ . This reduces to

$$(x + y)^2 + 6(x + y) - 8xy - 7 > 0. \quad (1)$$

But

$$1 = x^3 + y^3 = (x + y)(x^2 - xy + y^2),$$

which gives  $xy = ((x + y)^3 - 1)/3(x + y)$ . Substituting this in (1) and introducing  $x + y = t$ , the inequality takes the form

$$t^2 + 6t - \frac{8}{3} \frac{(t^3 - 1)}{t} - 7 > 0. \quad (2)$$

This may be simplified to  $-5t^3 + 18t^2 - 2t + 8 > 0$ . Equivalently

$$-(5t - 8)(t - 1)^2 > 0.$$

Thus we need to prove that  $5t < 8$ . Observe that  $(x + y)^3 > x^3 + y^3 = 1$ , so that  $t > 1$ . We also have

$$\left( \frac{x + y}{2} \right) \leq \frac{x^3 + y^3}{2} = \frac{1}{2}.$$

This shows that  $t^3 \leq 4$ . Thus

$$\left(\frac{5t}{8}\right)^3 \leq \frac{125 \times 4}{512} = \frac{500}{512} < 1.$$

Hence  $5t < 8$ , which proves the given inequality.

**3.** We write  $b^3 = c^3 - a^3$  and  $a^3 = c^3 - b^3$  so that

$$c - a = \frac{b^3}{c^2 - ca + a^2}, \quad c - b = \frac{a^3}{c^2 - cb + b^2}.$$

Thus the inequality reduces to

$$a^2 + b^2 - c^2 > 6 \frac{a^3 b^3}{(c^2 - ca + a^2)(c^2 - cb + b^2)}.$$

This simplifies(after some lengthy calculations) to

$$-c^6 - (a+b)c^5 - abc^4 + (a^3 + b^3)c^3 + (a^4 + a^3b + a^2b^2 + ab^3 + b^4)c^2 \\ (a^2b + ab^2 + a^3 + b^3)abc + (a^4b^2 - 6a^3b^3 + a^2b^4) > 0.$$

Substituting

$$c^3 = a^3 + b^3, \quad c^4 = c(a^3 + b^3), \quad c^5 = c^2(a^3 + b^3), \quad c^6 = (a^3 + b^3)^2,$$

the inequality further reduces to

$$a^2b^2(a^2 + b^2 + c^2 + ac + bc - 6ab) > 0.$$

Thus we need to prove that  $a^2 + b^2 + c^2 + ac + bc - 6ab > 0$ . Since  $a^2 + b^2 \geq 2ab$ , it is enough to prove that  $c^2 + c(a+b) - 4ab > 0$ . Multiplying this by  $c$  and using  $a^3 + b^3 = c^3$ , we need to prove that

$$a^3 + b^3 + c^2a + c^2b > 4abc.$$

Using AM-GM inequality to these 4 terms and using  $c > a, c > b$  we get

$$a^3 + b^3 + c^2a + c^2b > 4(a^3b^3c^2ac^2b)^{1/4} = 4abc,$$

which proves the inequality.

---

# INMO-2010 Problems and Solutions

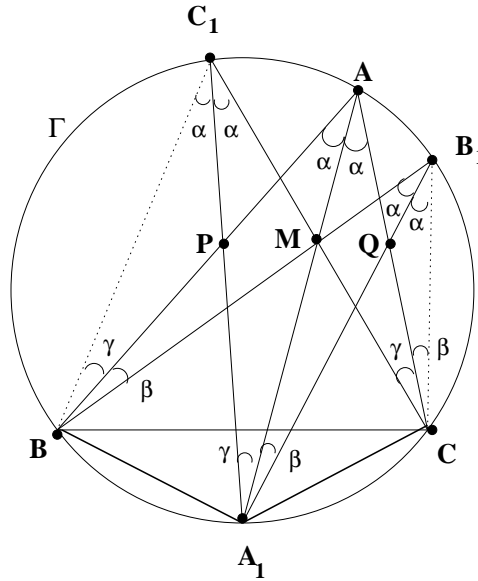
1. Let  $ABC$  be a triangle with circum-circle  $\Gamma$ . Let  $M$  be a point in the interior of triangle  $ABC$  which is also on the bisector of  $\angle A$ . Let  $AM$ ,  $BM$ ,  $CM$  meet  $\Gamma$  in  $A_1$ ,  $B_1$ ,  $C_1$  respectively. Suppose  $P$  is the point of intersection of  $A_1C_1$  with  $AB$ ; and  $Q$  is the point of intersection of  $A_1B_1$  with  $AC$ . Prove that  $PQ$  is parallel to  $BC$ .

**Solution:** Let  $A = 2\alpha$ . Then  $\angle A_1AC = \angle BAA_1 = \alpha$ . Thus

$$\angle A_1B_1C = \alpha = \angle BB_1A_1 = \angle A_1C_1C = \angle BC_1A_1.$$

We also have  $\angle B_1CQ = \angle AA_1B_1 = \beta$ , say. It follows that triangles  $MA_1B_1$  and  $QCB_1$  are similar and hence

$$\frac{QC}{MA_1} = \frac{B_1C}{B_1A_1}.$$



Similarly, triangles  $ACM$  and  $C_1A_1M$  are similar and we get

$$\frac{AC}{AM} = \frac{C_1A_1}{C_1M}.$$

Using the point  $P$ , we get similar ratios:

$$\frac{PB}{MA_1} = \frac{C_1B}{A_1C_1}, \quad \frac{AB}{AM} = \frac{A_1B_1}{MB_1}.$$

Thus,

$$\frac{QC}{PB} = \frac{A_1C_1 \cdot B_1C}{C_1B \cdot B_1A_1},$$

and

$$\begin{aligned} \frac{AC}{AB} &= \frac{MB_1 \cdot C_1A_1}{A_1B_1 \cdot C_1M} \\ &= \frac{MB_1}{C_1M} \frac{C_1A_1}{A_1B_1} = \frac{MB_1}{C_1M} \frac{C_1B \cdot QC}{PB \cdot B_1C}. \end{aligned}$$

However, triangles  $C_1BM$  and  $B_1CM$  are similar, which gives

$$\frac{B_1C}{C_1B} = \frac{MB_1}{MC_1}.$$

Putting this in the last expression, we get

$$\frac{AC}{AB} = \frac{QC}{PB}.$$

We conclude that  $PQ$  is parallel to  $BC$ .

2. Find all natural numbers  $n > 1$  such that  $n^2$  **does not** divide  $(n-2)!$ .

**Solution:** Suppose  $n = pqr$ , where  $p < q$  are primes and  $r > 1$ . Then  $p \geq 2$ ,  $q \geq 3$  and  $r \geq 2$ , not necessarily a prime. Thus we have

$$\begin{aligned} n-2 &\geq n-p = pqr - p \geq 5p > p, \\ n-2 &\geq n-q = q(pr-1) \geq 3q > q, \\ n-2 &\geq n-pr = pr(q-1) \geq 2pr > pr, \\ n-2 &\geq n-qr = qr(p-1) \geq qr. \end{aligned}$$

Observe that  $p, q, pr, qr$  are all distinct. Hence their product divides  $(n-2)!$ . Thus  $n^2 = p^2q^2r^2$  divides  $(n-2)!$  in this case. We conclude that either  $n = pq$  where  $p, q$  are distinct primes or  $n = p^k$  for some prime  $p$ .

**Case 1.** Suppose  $n = pq$  for some primes  $p, q$ , where  $2 < p < q$ . Then  $p \geq 3$  and  $q \geq 5$ . In this case

$$\begin{aligned} n-2 &> n-p = p(q-1) \geq 4p, \\ n-2 &> n-q = q(p-1) \geq 2q. \end{aligned}$$

Thus  $p, q, 2p, 2q$  are all distinct numbers in the set  $\{1, 2, 3, \dots, n-2\}$ . We see that  $n^2 = p^2q^2$  divides  $(n-2)!$ . We conclude that  $n = 2q$  for some prime  $q \geq 3$ . Note that  $n-2 = 2q-2 < 2q$  in this case so that  $n^2$  does not divide  $(n-2)!$ .

**Case 2.** Suppose  $n = p^k$  for some prime  $p$ . We observe that  $p, 2p, 3p, \dots, (p^{k-1}-1)p$  all lie in the set  $\{1, 2, 3, \dots, n-2\}$ . If  $p^{k-1}-1 \geq 2k$ , then there are at least  $2k$  multiples of  $p$  in the set  $\{1, 2, 3, \dots, n-2\}$ . Hence  $n^2 = p^{2k}$  divides  $(n-2)!$ . Thus  $p^{k-1}-1 < 2k$ .

If  $k \geq 5$ , then  $p^{k-1}-1 \geq 2^{k-1}-1 \geq 2k$ , which may be proved by an easy induction. Hence  $k \leq 4$ . If  $k = 1$ , we get  $n = p$ , a prime. If  $k = 2$ , then  $p-1 < 4$  so that  $p = 2$  or  $3$ ; we get  $n = 2^2 = 4$  or  $n = 3^2 = 9$ . For  $k = 3$ , we have  $p^2-1 < 6$  giving  $p = 2$ ;  $n = 2^3 = 8$  in this case. Finally,  $k = 4$  gives  $p^3-1 < 8$ . Again  $p = 2$  and  $n = 2^4 = 16$ . However  $n^2 = 2^8$  divides  $14!$  and hence is not a solution.

Thus  $n = p, 2p$  for some prime  $p$  or  $n = 8, 9$ . It is easy to verify that these satisfy the conditions of the problem.

3. Find all non-zero real numbers  $x, y, z$  which satisfy the system of equations:

$$\begin{aligned} (x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) &= xyz, \\ (x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) &= x^3y^3z^3. \end{aligned}$$

**Solution:** Since  $xyz \neq 0$ , We can divide the second relation by the first. Observe that

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$

holds for any  $x, y$ . Thus we get

$$(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) = x^2y^2z^2.$$

However, for any real numbers  $x, y$ , we have

$$x^2 - xy + y^2 \geq |xy|.$$

Since  $x^2 y^2 z^2 = |xy| |yz| |zx|$ , we get

$$|xy| |yz| |zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \geq |xy| |yz| |zx|.$$

This is possible only if

$$x^2 - xy + y^2 = |xy|, \quad y^2 - yz + z^2 = |yz|, \quad z^2 - zx + x^2 = |zx|,$$

hold simultaneously. However  $|xy| = \pm xy$ . If  $x^2 - xy + y^2 = -xy$ , then  $x^2 + y^2 = 0$  giving  $x = y = 0$ . Since we are looking for nonzero  $x, y, z$ , we conclude that  $x^2 - xy + y^2 = xy$  which is same as  $x = y$ . Using the other two relations, we also get  $y = z$  and  $z = x$ . The first equation now gives  $27x^6 = x^3$ . This gives  $x^3 = 1/27$  (since  $x \neq 0$ ), or  $x = 1/3$ . We thus have  $x = y = z = 1/3$ . These also satisfy the second relation, as may be verified.

4. How many 6-tuples  $(a_1, a_2, a_3, a_4, a_5, a_6)$  are there such that each of  $a_1, a_2, a_3, a_4, a_5, a_6$  is from the set  $\{1, 2, 3, 4\}$  and the six expressions

$$a_j^2 - a_j a_{j+1} + a_{j+1}^2$$

for  $j = 1, 2, 3, 4, 5, 6$  (where  $a_7$  is to be taken as  $a_1$ ) are all equal to one another?

**Solution:** Without loss of generality, we may assume that  $a_1$  is the largest among  $a_1, a_2, a_3, a_4, a_5, a_6$ . Consider the relation

$$a_1^2 - a_1 a_2 + a_2^2 = a_2^2 - a_2 a_3 + a_3^2.$$

This leads to

$$(a_1 - a_3)(a_1 + a_3 - a_2) = 0.$$

Observe that  $a_1 \geq a_2$  and  $a_3 > 0$  together imply that the second factor on the left side is positive. Thus  $a_1 = a_3 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}$ . Using this and the relation

$$a_3^2 - a_3 a_4 + a_4^2 = a_4^2 - a_4 a_5 + a_5^2,$$

we conclude that  $a_3 = a_5$  as above. Thus we have

$$a_1 = a_3 = a_5 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Let us consider the other relations. Using

$$a_2^2 - a_2 a_3 + a_3^2 = a_3^2 - a_3 a_4 + a_4^2,$$

we get  $a_2 = a_4$  or  $a_2 + a_4 = a_3 = a_1$ . Similarly, two more relations give either  $a_4 = a_6$  or  $a_4 + a_6 = a_5 = a_1$ ; and either  $a_6 = a_2$  or  $a_6 + a_2 = a_1$ . Let us give values to  $a_1$  and count the number of six-tuples in each case.

- (A) Suppose  $a_1 = 1$ . In this case all  $a_j$ 's are equal and we get only one six-tuple  $(1, 1, 1, 1, 1, 1)$ .
- (B) If  $a_1 = 2$ , we have  $a_3 = a_5 = 2$ . We observe that  $a_2 = a_4 = a_6 = 1$  or  $a_2 = a_4 = a_6 = 2$ . We get two more six-tuples:  $(2, 1, 2, 1, 2, 1)$ ,  $(2, 2, 2, 2, 2, 2)$ .
- (C) Taking  $a_1 = 3$ , we see that  $a_3 = a_5 = 3$ . In this case we get nine possibilities for  $(a_2, a_4, a_6)$ ;

$$(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).$$

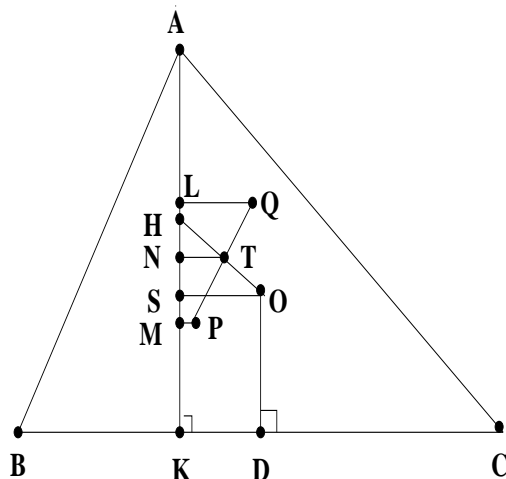
(D) In the case  $a_1 = 4$ , we have  $a_3 = a_5 = 4$  and

$$(a_2, a_4, a_6) = (2, 2, 2), (4, 4, 4), (1, 1, 1), (3, 3, 3), \\ (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 3, 3), (3, 1, 3), (3, 3, 1).$$

Thus we get  $1 + 2 + 9 + 10 = 22$  solutions. Since  $(a_1, a_3, a_5)$  and  $(a_2, a_4, a_6)$  may be interchanged, we get 22 more six-tuples. However there are 4 common among these, namely,  $(1, 1, 1, 1, 1, 1)$ ,  $(2, 2, 2, 2, 2, 2)$ ,  $(3, 3, 3, 3, 3, 3)$  and  $(4, 4, 4, 4, 4, 4)$ . Hence the total number of six-tuples is  $22 + 22 - 4 = 40$ .

5. Let  $ABC$  be an acute-angled triangle with altitude  $AK$ . Let  $H$  be its ortho-centre and  $O$  be its circum-centre. Suppose  $KOH$  is an acute-angled triangle and  $P$  its circum-centre. Let  $Q$  be the reflection of  $P$  in the line  $HO$ . Show that  $Q$  lies on the line joining the mid-points of  $AB$  and  $AC$ .

**Solution:** Let  $D$  be the mid-point of  $BC$ ;  $M$  that of  $HK$ ; and  $T$  that of  $OH$ . Then  $PM$  is perpendicular to  $HK$  and  $PT$  is perpendicular to  $OH$ . Since  $Q$  is the reflection of  $P$  in  $HO$ , we observe that  $P, T, Q$  are collinear, and  $PT = TQ$ . Let  $QL$ ,  $TN$  and  $OS$  be the perpendiculars drawn respectively from  $Q$ ,  $T$  and  $O$  on to the altitude  $AK$ . (See the figure.)



We have  $LN = NM$ , since  $T$  is the mid-point of  $QP$ ;  $HN = NS$ , since  $T$  is the mid-point of  $OH$ ; and  $HM = MK$ , as  $P$  is the circum-centre of  $KHO$ . We obtain

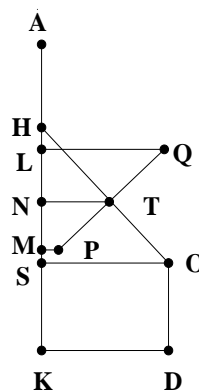
$$LH + HN = LN = NM = NS + SM,$$

which gives  $LH = SM$ . We know that  $AH = 2OD$ . Thus

$$AL = AH - LH = 2OD - LH = 2SK - SM = SK + (SK - SM) = SK + MK \\ = SK + HM = SK + HS + SM = SK + HS + LH = SK + LS = LK.$$

This shows that  $L$  is the mid-point of  $AK$  and hence lies on the line joining the midpoints of  $AB$  and  $AC$ . We observe that the line joining the mid-points of  $AB$  and  $AC$  is also perpendicular to  $AK$ . Since  $QL$  is perpendicular to  $AK$ , we conclude that  $Q$  also lies on the line joining the mid-points of  $AB$  and  $AC$ .

**Remark:** It may happen that  $H$  is above  $L$  as in the adjoining figure, but the result remains true here as well. We have  $HN = NS$ ,  $LN = NM$ , and  $HM = MK$  as earlier. Thus  $HN = HL + LN$  and  $NS = SM + NM$  give  $HL = SM$ . Now  $AL = AH + HL = 2OD + SM = 2SK + SM = SK + (SK + SM) = SK + MK = SK + HM = SK + HL + LM = SK + SM + LM = LK$ . The conclusion that  $Q$  lies on the line joining the mid-points of  $AB$  and  $AC$  follows as earlier.



6. Define a sequence  $\langle a_n \rangle_{n \geq 0}$  by  $a_0 = 0$ ,  $a_1 = 1$  and

$$a_n = 2a_{n-1} + a_{n-2},$$

for  $n \geq 2$ .

- (a) For every  $m > 0$  and  $0 \leq j \leq m$ , prove that  $2a_m$  divides  $a_{m+j} + (-1)^j a_{m-j}$ .  
(b) Suppose  $2^k$  divides  $n$  for some natural numbers  $n$  and  $k$ . Prove that  $2^k$  divides  $a_n$ .

**Solution:**

- (a) Consider  $f(j) = a_{m+j} + (-1)^j a_{m-j}$ ,  $0 \leq j \leq m$ , where  $m$  is a natural number. We observe that  $f(0) = 2a_m$  is divisible by  $2a_m$ . Similarly,

$$f(1) = a_{m+1} - a_{m-1} = 2a_m$$

is also divisible by  $2a_m$ . Assume that  $2a_m$  divides  $f(j)$  for all  $0 \leq j < l$ , where  $l \leq m$ . We prove that  $2a_m$  divides  $f(l)$ . Observe

$$\begin{aligned} f(l-1) &= a_{m+l-1} + (-1)^{l-1} a_{m-l+1}, \\ f(l-2) &= a_{m+l-2} + (-1)^{l-2} a_{m-l+2}. \end{aligned}$$

Thus we have

$$\begin{aligned} a_{m+l} &= 2a_{m+l-1} + a_{m+l-2} \\ &= 2f(l-1) - 2(-1)^{l-1} a_{m-l+1} + f(l-2) - (-1)^{l-2} a_{m-l+2} \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} (a_{m-l+2} - 2a_{m-l+1}) \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} a_{m-l}. \end{aligned}$$

This gives

$$f(l) = 2f(l-1) + f(l-2).$$

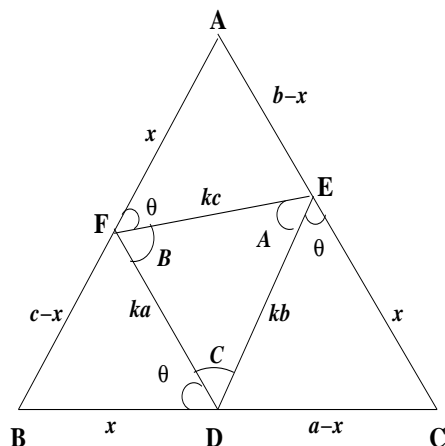
By induction hypothesis  $2a_m$  divides  $f(l-1)$  and  $f(l-2)$ . Hence  $2a_m$  divides  $f(l)$ . We conclude that  $2a_m$  divides  $f(j)$  for  $0 \leq j \leq m$ .

- (b) We see that  $f(m) = a_{2m}$ . Hence  $2a_m$  divides  $a_{2m}$  for all natural numbers  $m$ . Let  $n = 2^k l$  for some  $l \geq 1$ . Taking  $m = 2^{k-1} l$ , we see that  $2a_m$  divides  $a_n$ . Using an easy induction, we conclude that  $2^k a_l$  divides  $a_n$ . In particular  $2^k$  divides  $a_n$ .

## Problems and Solutions, INMO-2011

1. Let  $D, E, F$  be points on the sides  $BC, CA, AB$  respectively of a triangle  $ABC$  such that  $BD = CE = AF$  and  $\angle BDF = \angle CED = \angle AFE$ . Prove that  $ABC$  is equilateral.

**Solution 1:**



Let  $BD = CE = AF = x$ ;  $\angle BDF = \angle CED = \angle AFE = \theta$ . Note that  $\angle AFD = B + \theta$ , and hence  $\angle DFE = B$ . Similarly,  $\angle EDF = C$  and  $\angle FED = A$ . Thus the triangle  $EFD$  is similar to  $ABC$ . We may take  $FD = ka$ ,  $DE = kb$  and  $EF = kc$ , for some positive real constant  $k$ . Applying sine rule to triangle  $BFD$ , we obtain

$$\frac{c-x}{\sin \theta} = \frac{ka}{\sin B} = \frac{2Rka}{b},$$

where  $R$  is the circum-radius of  $ABC$ . Thus we get  $2Rk \sin \theta = b(c-x)/a$ . Similarly, we obtain  $2Rk \sin \theta = c(a-x)/b$  and  $2Rk \sin \theta = a(b-x)/c$ . We therefore get

$$\frac{b(c-x)}{a} = \frac{c(a-x)}{b} = \frac{a(b-x)}{c}. \quad (1)$$

If some two sides are equal, say,  $a = b$ , then  $a(c-x) = c(a-x)$  giving  $a = c$ ; we get  $a = b = c$  and  $ABC$  is equilateral. Suppose no two sides of  $ABC$  are equal. We may assume  $a$  is the least. Since (1) is cyclic in  $a, b, c$ , we have to consider two cases:  $a < b < c$  and  $a < c < b$ .

**Case 1.**  $a < b < c$ .

In this case  $a < c$  and hence  $b(c-x) < a(b-x)$ , from (1). Since  $b > a$  and  $c-x > b-x$ , we get  $b(c-x) > a(b-x)$ , which is a contradiction.

**Case 2.**  $a < c < b$ .

We may write (1) in the form

$$\frac{(c-x)}{a/b} = \frac{(a-x)}{b/c} = \frac{(b-x)}{c/a}. \quad (2)$$

Now  $a < c$  gives  $a-x < c-x$  so that  $\frac{b}{c} < \frac{a}{b}$ . This gives  $b^2 < ac$ . But  $b > a$  and  $b > c$ , so that  $b^2 > ac$ , which again leads to a contradiction.

Thus Case 1 and Case 2 cannot occur. We conclude that  $a = b = c$ .

**Solution 2.** We write (1) in the form (2), and start from there. The case of two equal sides is dealt as in Solution 1. We assume no two sides are equal. Using ratio properties in (2), we obtain

$$\frac{a-b}{(ab-c^2)/ca} = \frac{b-c}{(bc-a^2)/ab}.$$

This may be written as  $c(a-b)(bc-a^2) = b(b-c)(ab-c^2)$ . Further simplification gives  $ab^3 + bc^3 + ca^3 = abc(a+b+c)$ . This may be further written in the form

$$ab^2(b-c) + bc^2(c-a) + ca^2(a-b) = 0. \quad (3)$$

If  $a < b < c$ , we write (3) in the form

$$0 = ab^2(b-c) + bc^2(c-b+b-a) + ca^2(a-b) = b(c-b)(c^2-ab) + c(b-a)(bc-a^2).$$

Since  $c > b$ ,  $c^2 > ab$ ,  $b > a$  and  $bc > a^2$ , this is impossible. If  $a < c < b$ , we write (3), as in previous case, in the form

$$0 = a(b-c)(b^2-ca) + c(c-a)(bc-a^2),$$

which again is impossible.

One can also use inequalities: we can show that  $ab^3 + bc^3 + ca^3 \geq abc(a+b+c)$ , and equality holds if and only if  $a = b = c$ . Here are some ways of deriving it:

(i) We can write the inequality in the form

$$\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} \geq a + b + c.$$

Adding  $a + b + c$  both sides, this takes the form

$$\frac{b^2}{c} + c + \frac{c^2}{a} + a + \frac{a^2}{b} + b \geq 2(a + b + c).$$

But AM-GM inequality gives

$$\frac{b^2}{c} + c \geq 2b, \quad \frac{c^2}{a} + a \geq 2a, \quad \frac{a^2}{b} + b \geq 2a.$$

Hence the inequality follows and equality holds if and only if  $a = b = c$ .

(ii) Again we write the inequality in the form

$$\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} \geq a + b + c.$$

We use  $b/c$  with weight  $b$ ,  $c/a$  with weight  $c$  and  $a/b$  with weight  $a$ , and apply weighted AM-HM inequality:

$$b \cdot \frac{b}{c} + c \cdot \frac{c}{a} + a \cdot \frac{a}{b} \geq \frac{(a + b + c)^2}{b \cdot \frac{c}{b} + c \cdot \frac{a}{c} + a \cdot \frac{b}{a}},$$

which reduces to  $a + b + c$ . Again equality holds if and only if  $a = b = c$ .

**Solution 3.** Here is a pure geometric solution given by a student. Consider the triangle  $BDF$ ,  $CED$  and  $AFE$  with  $BD$ ,  $CE$  and  $AF$  as bases. The sides  $DF$ ,  $ED$  and  $FE$  make equal angles  $\theta$  with the bases of respective triangles. If  $B \geq C \geq A$ , then it is easy to see that  $FD \geq DE \geq EF$ . Now using the triangle  $FDE$ , we see that  $B \geq C \geq A$  gives  $DE \geq EF \geq FD$ . Combining, you get  $FD = DE = EF$  and hence  $A = B = C = 60^\circ$ .

2. Call a natural number  $n$  *faithful*, if there exist natural numbers  $a < b < c$  such that  $a$  divides  $b$ ,  $b$  divides  $c$  and  $n = a + b + c$ .

(i) Show that all but a finite number of natural numbers are faithful.

(ii) Find the sum of all natural numbers which are **not** faithful.

**Solution 1:** Suppose  $n \in \mathbb{N}$  is faithful. Let  $k \in \mathbb{N}$  and consider  $kn$ . Since  $n = a + b + c$ , with  $a > b > c$ ,  $c|b$  and  $b|a$ , we see that  $kn = ka + kb + kc$  which shows that  $kn$  is faithful.

Let  $p > 5$  be a prime. Then  $p$  is odd and  $p = (p - 3) + 2 + 1$  shows that  $p$  is faithful. If  $n \in \mathbb{N}$  contains a prime factor  $p > 5$ , then the above observation shows that  $n$  is faithful. This shows that a number which is not faithful must be of the form  $2^\alpha 3^\beta 5^\gamma$ . We also observe that  $2^4 = 16 = 12 + 3 + 1$ ,  $3^2 = 9 = 6 + 2 + 1$  and  $5^2 = 25 = 22 + 2 + 1$ , so that  $2^4$ ,  $3^2$  and  $5^2$  are faithful. Hence  $n \in \mathbb{N}$  is also faithful if it contains a factor of the form  $2^\alpha$  where  $\alpha \geq 4$ ; a factor of the form  $3^\beta$  where  $\beta \geq 2$ ; or a factor of the form  $5^\gamma$  where  $\gamma \geq 2$ . Thus the numbers which are not faithful are of the form  $2^\alpha 3^\beta 5^\gamma$ , where  $\alpha \leq 3$ ,  $\beta \leq 1$  and  $\gamma \leq 1$ . We may enumerate all such numbers:

$$1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.$$

Among these  $120 = 112 + 7 + 1$ ,  $60 = 48 + 8 + 4$ ,  $40 = 36 + 3 + 1$ ,  $30 = 18 + 9 + 3$ ,  $20 = 12 + 6 + 2$ ,  $15 = 12 + 2 + 1$ , and  $10 = 6 + 3 + 1$ . It is easy to check that the other numbers cannot be written in the required form. Hence the only numbers which are not faithful are

$$1, 2, 3, 4, 5, 6, 8, 12, 24.$$

Their sum is 65.

**Solution 2:** If  $n = a + b + c$  with  $a < b < c$  is faithful, we see that  $a \geq 1$ ,  $b \geq 2$  and  $c \geq 4$ . Hence  $n \geq 7$ . Thus  $1, 2, 3, 4, 5, 6$  are not faithful. As observed earlier,  $kn$  is faithful whenever

$n$  is. We also notice that for odd  $n \geq 7$ , we can write  $n = 1 + 2 + (n - 3)$  so that all odd  $n \geq 7$  are faithful. Consider  $2n, 4n, 8n$ , where  $n \geq 7$  is odd. By observation, they are all faithful. Let us list a few of them:

$$\begin{aligned} 2n &: 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, \dots \\ 4n &: 28, 36, 44, 52, 60, 68, \dots \\ 8n &: 56, 72, \dots, \end{aligned}$$

We observe that  $16 = 12 + 3 + 1$  and hence it is faithful. Thus all multiples of 16 are also faithful. Thus we see that 16, 32, 48, 64, ... are faithful. Any even number which is not a multiple of 16 must be either an odd multiple of 2, or that of 4, or that of 8. Hence, the only numbers not covered by this process are 8, 10, 12, 20, 24, 40. Of these, we see that

$$10 = 1 + 3 + 6, \quad 20 = 2 \times 10, \quad 40 = 4 \times 10,$$

so that 10, 20, 40 are faithful. Thus the only numbers which are not faithful are

$$1, 2, 3, 4, 5, 6, 8, 12, 24.$$

Their sum is 65.

3. Consider two polynomials  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  and  $Q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$  with integer coefficients such that  $a_n - b_n$  is a prime,  $a_{n-1} = b_{n-1}$  and  $a_n b_0 - a_0 b_n \neq 0$ . Suppose there exists a rational number  $r$  such that  $P(r) = Q(r) = 0$ . Prove that  $r$  is an integer.

**Solution:** Let  $r = u/v$  where  $\gcd(u, v) = 1$ . Then we get

$$\begin{aligned} a_n u^n + a_{n-1} u^{n-1} v + \dots + a_1 u v^{n-1} + a_0 v^n &= 0, \\ b_n u^n + b_{n-1} u^{n-1} v + \dots + b_1 u v^{n-1} + b_0 v^n &= 0. \end{aligned}$$

Subtraction gives

$$(a_n - b_n)u^n + (a_{n-2} - b_{n-2})u^{n-2}v^2 + \dots + (a_1 - b_1)uv^{n-1} + (a_0 - b_0)v^n = 0,$$

since  $a_{n-1} = b_{n-1}$ . This shows that  $v$  divides  $(a_n - b_n)u^n$  and hence it divides  $a_n - b_n$ . Since  $a_n - b_n$  is a prime, either  $v = 1$  or  $v = a_n - b_n$ . Suppose the latter holds. The relation takes the form

$$u^n + (a_{n-2} - b_{n-2})u^{n-2}v + \dots + (a_1 - b_1)uv^{n-2} + (a_0 - b_0)v^{n-1} = 0.$$

(Here we have divided through-out by  $v$ .) If  $n > 1$ , this forces  $v|u$ , which is impossible since  $\gcd(v, u) = 1$  ( $v > 1$  since it is equal to the prime  $a_n - b_n$ ). If  $n = 1$ , then we get two equations:

$$\begin{aligned} a_1 u + a_0 v &= 0, \\ b_1 u + b_0 v &= 0. \end{aligned}$$

This forces  $a_1 b_0 - a_0 b_1 = 0$  contradicting  $a_n b_0 - a_0 b_n \neq 0$ . (Note: The condition  $a_n b_0 - a_0 b_n \neq 0$  is extraneous. The condition  $a_{n-1} = b_{n-1}$  forces that for  $n = 1$ , we have  $a_0 = b_0$ . Thus we obtain, after subtraction

$$(a_1 - b_1)u = 0.$$

This implies that  $u = 0$  and hence  $r = 0$  is an integer.)

4. Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium.

**Solution 1:** Suppose four distinct points  $P, Q, R, S$  (in that order on the circle) among these five are such that  $\widehat{PQ} = \widehat{RS}$ . Then  $PQRS$  is an isosceles trapezium, with  $PS \parallel QR$ . We use this in our argument.

- If four of the five points chosen are adjacent, then we are through as observed earlier. (In this case four points  $A, B, C, D$  are such that  $\widehat{AB} = \widehat{BC} = \widehat{CD}$ .) See Fig 1.

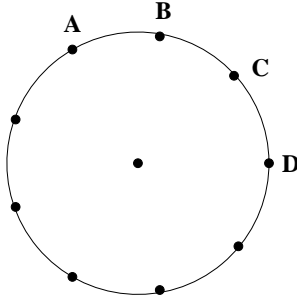


Fig 1.

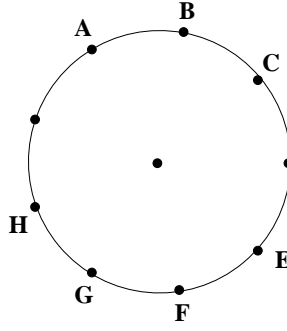


Fig 2.

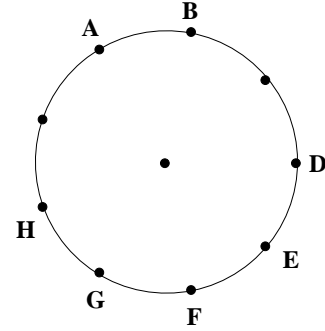


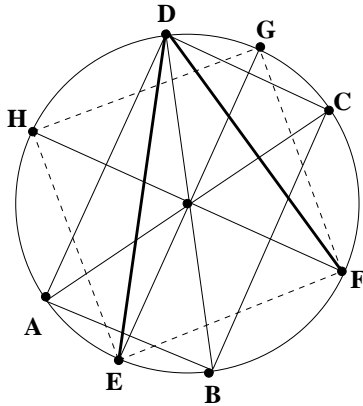
Fig 3.

- Suppose only three of the vertices are adjacent, say  $A, B, C$  (see Fig 2.) Then the remaining two must be among  $E, F, G, H$ . If these two are adjacent vertices, we can pair them with  $A, B$  or  $B, C$  to get equal arcs. If they are not adjacent, then they must be either  $E, G$  or  $F, H$  or  $E, H$ . In the first two cases, we can pair them with  $A, C$  to get equal arcs. In the last case, we observe that  $\widehat{HA} = \widehat{CE}$  and  $AHEC$  is an isosceles trapezium.
  - Suppose only two among the five are adjacent, say  $A, B$ . Then the remaining three are among  $D, E, F, G, H$ . (See Fig 3.) If any two of these are adjacent, we can combine them with  $A, B$  to get equal arcs. If no two among these three vertices are adjacent, then they must be  $D, F, H$ . In this case  $\widehat{HA} = \widehat{BD}$  and  $AHDB$  is an isosceles trapezium.
- Finally, if we choose 5 among the 9 vertices of a regular nine-sided polygon, then some two must be adjacent. Thus any choice of 5 among 9 must fall in to one of the above three possibilities.

**Solution 2:** Here is another solution used by many students. Suppose you join the vertices of the nine-sided regular polygon. You get  $\binom{9}{2} = 36$  line segments. All these fall in to 9 sets of parallel lines. Now using any 5 points, you get  $\binom{5}{2} = 10$  line segments. By pigeon-hole principle, two of these must be parallel. But, these parallel lines determine a trapezium.

5. Let  $ABCD$  be a quadrilateral inscribed in a circle  $\Gamma$ . Let  $E, F, G, H$  be the midpoints of the arcs  $AB, BC, CD, DA$  of the circle  $\Gamma$ . Suppose  $AC \cdot BD = EG \cdot FH$ . Prove that  $AC, BD, EG, FH$  are concurrent.

**Solution:**



Let  $R$  be the radius of the circle  $\Gamma$ . Observe that  $\angle EDF = \frac{1}{2}\angle D$ . Hence  $EF = 2R \sin \frac{D}{2}$ . Similarly,  $HG = 2R \sin \frac{B}{2}$ . But  $\angle B = 180^\circ - \angle D$ . Thus  $HG = 2R \cos \frac{D}{2}$ . We hence get

$$EF \cdot GH = 4R^2 \sin \frac{D}{2} \cos \frac{D}{2} = 2R^2 \sin D = R \cdot AC.$$

Similarly, we obtain  $EH \cdot FG = R \cdot BD$ .

Therefore

$$R(AC + BD) = EF \cdot GH + EH \cdot FG = EG \cdot FH,$$

by Ptolemy's theorem. By the given hypothesis, this gives  $R(AC + BD) = AC \cdot BD$ . Thus

$$AC \cdot BD = R(AC + BD) \geq 2R\sqrt{AC \cdot BD},$$

using AM-GM inequality. This implies that  $AC \cdot BD \geq 4R^2$ . But  $AC$  and  $BD$  are the chords of  $\Gamma$ , so that  $AC \leq 2R$  and  $BD \leq 2R$ . We obtain  $AC \cdot BD \leq 4R^2$ . It follows that  $AC \cdot BD = 4R^2$ , implying that  $AC = BD = 2R$ . Thus  $AC$  and  $BD$  are two diameters of  $\Gamma$ . Using  $EG \cdot FH = AC \cdot BD$ , we conclude that  $EG$  and  $FH$  are also two diameters of  $\Gamma$ . Hence  $AC, BD, EG$  and  $FH$  all pass through the centre of  $\Gamma$ .

6. Find all functions  $f : \mathbf{R} \rightarrow \mathbf{R}$  such that

$$f(x+y)f(x-y) = (f(x) + f(y))^2 - 4x^2f(y), \quad (1)$$

for all  $x, y \in \mathbf{R}$ , where  $\mathbf{R}$  denotes the set of all real numbers.

**Solution 1.:** Put  $x = y = 0$ ; we get  $f(0)^2 = 4f(0)^2$  and hence  $f(0) = 0$ .

Put  $x = y$ : we get  $4f(x)^2 - 4x^2f(x) = 0$  for all  $x$ . Hence for each  $x$ , either  $f(x) = 0$  or  $f(x) = x^2$ .

Suppose  $f(x) \neq 0$ . Then we can find  $x_0 \neq 0$  such that  $f(x_0) \neq 0$ . Then  $f(x_0) = x_0^2 \neq 0$ . Assume that there exists some  $y_0 \neq 0$  such that  $f(y_0) = 0$ . Then

$$f(x_0 + y_0)f(x_0 - y_0) = f(x_0)^2.$$

Now  $f(x_0 + y_0)f(x_0 - y_0) = 0$  or  $f(x_0 + y_0)f(x_0 - y_0) = (x_0 + y_0)^2(x_0 - y_0)^2$ . If  $f(x_0 + y_0)f(x_0 - y_0) = 0$ , then  $f(x_0) = 0$ , a contradiction. Hence it must be the latter so that

$$(x_0^2 - y_0^2)^2 = x_0^4.$$

This reduces to  $y_0^2(y_0^2 - 2x_0^2) = 0$ . Since  $y_0 \neq 0$ , we get  $y_0 = \pm\sqrt{2}x_0$ .

Suppose  $y_0 = \sqrt{2}x_0$ . Put  $x = \sqrt{2}x_0$  and  $y = x_0$  in (1); we get

$$f((\sqrt{2}+1)x_0)f((\sqrt{2}-1)x_0) = (f(\sqrt{2}x_0) + f(x_0))^2 - 4(2x_0^2)f(x_0).$$

But  $f(\sqrt{2}x_0) = f(y_0) = 0$ . Thus we get

$$\begin{aligned} f((\sqrt{2}+1)x_0)f((\sqrt{2}-1)x_0) &= f(x_0)^2 - 8x_0^2f(x_0) \\ &= x_0^4 - 8x_0^4 = -7x_0^4. \end{aligned}$$

Now if LHS is equal to 0, we get  $x_0 = 0$ , a contradiction. Otherwise LHS is equal to  $(\sqrt{2}+1)^2(\sqrt{2}-1)^2x_0^4$  which reduces to  $x_0^4$ . We obtain  $x_0^4 = -7x_0^4$  and this forces again  $x_0 = 0$ . Hence there is no  $y \neq 0$  such that  $f(y) = 0$ . We conclude that  $f(x) = x^2$  for all  $x$ .

Thus there are two solutions:  $f(x) = 0$  for all  $x$  or  $f(x) = x^2$ , for all  $x$ . It is easy to verify that both these satisfy the functional equation.

**Solution 2:** As earlier, we get  $f(0) = 0$ . Putting  $x = 0$ , we will also get

$$f(y)(f(y) - f(-y)) = 0.$$

As earlier, we may conclude that either  $f(y) = 0$  or  $f(y) = f(-y)$  for each  $y \in \mathbf{R}$ . Replacing  $y$  by  $-y$ , we may also conclude that  $f(-y)(f(-y) - f(y)) = 0$ . If  $f(y) = 0$  and  $f(-y) \neq 0$  for some  $y$ , then we must have  $f(-y) = f(y) = 0$ , a contradiction. Hence either  $f(y) = f(-y) = 0$  or  $f(y) = f(-y)$  for each  $y$ . This forces  $f$  is an even function.

Taking  $y = 1$  in (1), we get

$$f(x+1)f(x-1) = (f(x) + f(1))^2 - 4x^2f(1).$$

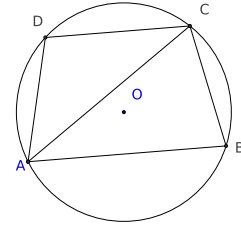
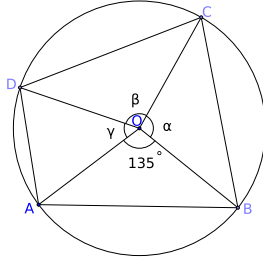
Replacing  $y$  by  $x$  and  $x$  by 1, you also get

$$f(1+x)f(1-x) = (f(1) + f(x))^2 - 4f(x).$$

Comparing these two using the even nature of  $f$ , we get  $f(x) = cx^2$ , where  $c = f(1)$ . Putting  $x = y = 1$  in (1), you get  $4c^2 - 4c = 0$ . Hence  $c = 0$  or 1. We get  $f(x) = 0$  for all  $x$  or  $f(x) = x^2$  for all  $x$ .

## Problems and Solutions: INMO-2012

1. Let  $ABCD$  be a quadrilateral inscribed in a circle. Suppose  $AB = \sqrt{2 + \sqrt{2}}$  and  $AB$  subtends  $135^\circ$  at the centre of the circle. Find the maximum possible area of  $ABCD$ .



**Solution:** Let  $O$  be the centre of the circle in which  $ABCD$  is inscribed and let  $R$  be its radius. Using cosine rule in triangle  $AOB$ , we have

$$2 + \sqrt{2} = 2R^2(1 - \cos 135^\circ) = R^2(2 + \sqrt{2}).$$

Hence  $R = 1$ .

Consider quadrilateral  $ABCD$  as in the second figure above. Join  $AC$ . For  $[ADC]$  to be maximum, it is clear that  $D$  should be the mid-point of the arc  $AC$  so that its distance from the segment  $AC$  is maximum. Hence  $AD = DC$  for  $[ABCD]$  to be maximum. Similarly, we conclude that  $BC = CD$ . Thus  $BC = CD = DA$  which fixes the quadrilateral  $ABCD$ . Therefore each of the sides  $BC$ ,  $CD$ ,  $DA$  subtends equal angles at the centre  $O$ .

Let  $\angle BOC = \alpha$ ,  $\angle COD = \beta$  and  $\angle DOA = \gamma$ . Observe that

$$[ABCD] = [AOB] + [BOC] + [COD] + [DOA] = \frac{1}{2} \sin 135^\circ + \frac{1}{2} (\sin \alpha + \sin \beta + \sin \gamma).$$

Now  $[ABCD]$  has maximum area if and only if  $\alpha = \beta = \gamma = (360^\circ - 135^\circ)/3 = 75^\circ$ . Thus

$$[ABCD] = \frac{1}{2} \sin 135^\circ + \frac{3}{2} \sin 75^\circ = \frac{1}{2} \left( \frac{1}{\sqrt{2}} + 3 \frac{\sqrt{3} + 1}{2\sqrt{2}} \right) = \frac{5 + 3\sqrt{3}}{4\sqrt{2}}.$$

Alternatively, we can use Jensen's inequality. Observe that  $\alpha, \beta, \gamma$  are all less than  $180^\circ$ . Since  $\sin x$  is concave on  $(0, \pi)$ , Jensen's inequality gives

$$\frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \leq \sin \left( \frac{\alpha + \beta + \gamma}{3} \right) = \sin 75^\circ.$$

Hence

$$[ABCD] \leq \frac{1}{2\sqrt{2}} + \frac{3}{2} \sin 75^\circ = \frac{5 + 3\sqrt{3}}{4\sqrt{2}},$$

with equality if and only if  $\alpha = \beta = \gamma = 75^\circ$ .

2. Let  $p_1 < p_2 < p_3 < p_4$  and  $q_1 < q_2 < q_3 < q_4$  be two sets of prime numbers such that  $p_4 - p_1 = 8$  and  $q_4 - q_1 = 8$ . Suppose  $p_1 > 5$  and  $q_1 > 5$ . Prove that 30 divides  $p_1 - q_1$ .

**Solution:** Since  $p_4 - p_1 = 8$ , and no prime is even, we observe that  $\{p_1, p_2, p_3, p_4\}$  is a subset of  $\{p_1, p_1 + 2, p_1 + 4, p_1 + 6, p_1 + 8\}$ . Moreover  $p_1$  is larger than 3. If  $p_1 \equiv 1 \pmod{3}$ , then  $p_1 + 2$  and  $p_1 + 8$  are divisible by 3. Hence we do not get 4 primes in the set  $\{p_1, p_1 + 2, p_1 + 4, p_1 + 6, p_1 + 8\}$ . Thus  $p_1 \equiv 2 \pmod{3}$  and  $p_1 + 4$  is not a prime. We get  $p_2 = p_1 + 2, p_3 = p_1 + 6, p_4 = p_1 + 8$ .

Consider the remainders of  $p_1, p_1 + 2, p_1 + 6, p_1 + 8$  when divided by 5. If  $p_1 \equiv 2 \pmod{5}$ , then  $p_1 + 8$  is divisible by 5 and hence is not a prime. If  $p_1 \equiv 3 \pmod{5}$ , then  $p_1 + 2$  is divisible by 5. If  $p_1 \equiv 4 \pmod{5}$ , then  $p_1 + 6$  is divisible by 5. Hence the only possibility is  $p_1 \equiv 1 \pmod{5}$ .

Thus we see that  $p_1 \equiv 1 \pmod{2}$ ,  $p_1 \equiv 2 \pmod{3}$  and  $p_1 \equiv 1 \pmod{5}$ . We conclude that  $p_1 \equiv 11 \pmod{30}$ .

Similarly  $q_1 \equiv 11 \pmod{30}$ . It follows that 30 divides  $p_1 - q_1$ .

3. Define a sequence  $\langle f_0(x), f_1(x), f_2(x), \dots \rangle$  of functions by

$$f_0(x) = 1, \quad f_1(x) = x, \quad (f_n(x))^2 - 1 = f_{n+1}(x)f_{n-1}(x), \quad \text{for } n \geq 1.$$

Prove that each  $f_n(x)$  is a polynomial with integer coefficients.

**Solution:** Observe that

$$f_n^2(x) - f_{n-1}(x)f_{n+1}(x) = 1 = f_{n-1}^2(x) - f_{n-2}(x)f_n(x).$$

This gives

$$f_n(x)(f_n(x) + f_{n-2}(x)) = f_{n-1}(f_{n-1}(x) + f_{n+1}(x)).$$

We write this as

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_{n-2}(x) + f_n(x)}{f_{n-1}(x)}.$$

Using induction, we get

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_0(x) + f_2(x)}{f_1(x)}.$$

Observe that

$$f_2(x) = \frac{f_1^2(x) - 1}{f_0(x)} = x^2 - 1.$$

Hence

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{1 + (x^2 - 1)}{x} = x.$$

Thus we obtain

$$f_{n+1}(x) = xf_n(x) - f_{n-1}(x).$$

Since  $f_0(x)$ ,  $f_1(x)$  and  $f_2(x)$  are polynomials with integer coefficients, induction again shows that  $f_n(x)$  is a polynomial with integer coefficients.

**Note:** We can get  $f_n(x)$  explicitly:

$$f_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \binom{n-3}{3}x^{n-6} + \dots$$

4. Let  $ABC$  be a triangle. An interior point  $P$  of  $ABC$  is said to be **good** if we can find exactly 27 rays emanating from  $P$  intersecting the sides of the triangle  $ABC$  such that the triangle is divided by these rays into 27 smaller triangles of equal area. Determine the number of **good** points for a given triangle  $ABC$ .

**Solution:** Let  $P$  be a good point. Let  $l, m, n$  be respectively the number of parts the sides  $BC$ ,  $CA$ ,  $AB$  are divided by the rays starting from  $P$ . Note that a ray must pass through each of the vertices the triangle  $ABC$ ; otherwise we get some quadrilaterals.

Let  $h_1$  be the distance of  $P$  from  $BC$ . Then  $h_1$  is the height for all the triangles with their bases on  $BC$ . Equality of areas implies that all these bases have equal length. If we denote this by  $x$ , we get  $lx = a$ . Similarly, taking  $y$  and  $z$  as the lengths of the bases of triangles on  $CA$  and  $AB$  respectively, we get  $my = b$  and  $nz = c$ . Let  $h_2$  and  $h_3$  be the distances of  $P$  from  $CA$  and  $AB$  respectively. Then

$$h_1x = h_2y = h_3z = \frac{2\Delta}{27},$$

where  $\Delta$  denotes the area of the triangle  $ABC$ . These lead to

$$h_1 = \frac{2\Delta}{27} \frac{l}{a}, \quad h_2 = \frac{2\Delta}{27} \frac{m}{b}, \quad h_3 = \frac{2\Delta}{27} \frac{n}{c}.$$

But

$$\frac{2\Delta}{a} = h_a, \quad \frac{2\Delta}{b} = h_b, \quad \frac{2\Delta}{c} = h_c.$$

Thus we get

$$\frac{h_1}{h_a} = \frac{l}{27}, \quad \frac{h_2}{h_b} = \frac{m}{27}, \quad \frac{h_3}{h_c} = \frac{n}{27}.$$

However, we also have

$$\frac{h_1}{h_a} = \frac{[PBC]}{\Delta}, \quad \frac{h_2}{h_b} = \frac{[PCA]}{\Delta}, \quad \frac{h_3}{h_c} = \frac{[PAB]}{\Delta}.$$

Adding these three relations,

$$\frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

Thus

$$\frac{l}{27} + \frac{m}{27} + \frac{n}{27} = \frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

We conclude that  $l + m + n = 27$ . Thus every **good** point  $P$  determines a partition  $(l, m, n)$  of 27 such that there are  $l, m, n$  equal segments respectively on  $BC, CA, AB$ .

Conversely, take any partition  $(l, m, n)$  of 27. Divide  $BC, CA, AB$  respectively in to  $l, m, n$  equal parts. Define

$$h_1 = \frac{2l\Delta}{27a}, \quad h_2 = \frac{2m\Delta}{27b}.$$

Draw a line parallel to  $BC$  at a distance  $h_1$  from  $BC$ ; draw another line parallel to  $CA$  at a distance  $h_2$  from  $CA$ . Both lines are drawn such that they intersect at a point  $P$  inside the triangle  $ABC$ . Then

$$[PBC] = \frac{1}{2}ah_1 = \frac{l\Delta}{27}, \quad [PCA] = \frac{m\Delta}{27}.$$

Hence

$$[PAB] = \frac{n\Delta}{27}.$$

This shows that the distance of  $P$  from  $AB$  is

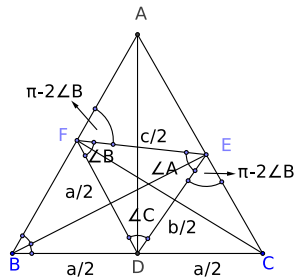
$$h_3 = \frac{2n\Delta}{27c}.$$

Therefore each triangle with base on  $CA$  has area  $\frac{\Delta}{27}$ . We conclude that all the triangles which partitions  $ABC$  have equal areas. Hence  $P$  is a **good** point.

Thus the number of **good** points is equal to the number of positive integral solutions of the equation  $l + m + n = 27$ . This is equal to

$$\binom{26}{2} = 325.$$

5. Let  $ABC$  be an acute-angled triangle, and let  $D, E, F$  be points on  $BC, CA, AB$  respectively such that  $AD$  is the median,  $BE$  is the internal angle bisector and  $CF$  is the altitude. Suppose  $\angle FDE = \angle C$ ,  $\angle DEF = \angle A$  and  $\angle EFD = \angle B$ . Prove that  $ABC$  is equilateral.



**Solution:** Since  $\triangle BFC$  is right-angled at  $F$ , we have  $FD = BD = CD = a/2$ . Hence  $\angle BFD = \angle B$ . Since  $\angle EFD = \angle B$ , we have  $\angle AFE = \pi - 2\angle B$ . Since  $\angle DEF = \angle A$ , we also get  $\angle CED = \pi - 2\angle B$ . Applying sine rule in  $\triangle DEF$ , we have

$$\frac{DF}{\sin A} = \frac{FE}{\sin C} = \frac{DE}{\sin B}.$$

Thus we get  $FE = c/2$  and  $DE = b/2$ . Sine rule in  $\triangle CED$  gives

$$\frac{DE}{\sin C} = \frac{CD}{\sin(\pi - 2B)}.$$

Thus  $(b/\sin C) = (a/2 \sin B \cos B)$ . Solving for  $\cos B$ , we have

$$\cos B = \frac{a \sin c}{2b \sin B} = \frac{ac}{2b^2}.$$

Similarly, sine rule in  $\triangle AEF$  gives

$$\frac{EF}{\sin A} = \frac{AE}{\sin(\pi - 2B)}.$$

This gives (since  $AE = bc/(a + c)$ ), as earlier,

$$\cos B = \frac{a}{a + c}.$$

Comparing the two values of  $\cos B$ , we get  $2b^2 = c(a + c)$ . We also have

$$c^2 + a^2 - b^2 = 2ca \cos B = \frac{2a^2c}{a + c}.$$

Thus

$$4a^2c = (a + c)(2c^2 + 2a^2 - 2b^2) = (a + c)(2c^2 + 2a^2 - c(a + c)).$$

This reduces to  $2a^3 - 3a^2c + c^3 = 0$ . Thus  $(a - c)^2(2a + c) = 0$ . We conclude that  $a = c$ . Finally

$$2b^2 = c(a + c) = 2c^2.$$

We thus get  $b = c$  and hence  $a = c = b$ . This shows that  $\triangle ABC$  is equilateral.

6. Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be a function satisfying  $f(0) \neq 0$ ,  $f(1) = 0$  and

(i)  $f(xy) + f(x)f(y) = f(x) + f(y);$

(ii)  $(f(x - y) - f(0))f(x)f(y) = 0,$

for all  $x, y \in \mathbb{Z}$ , simultaneously.

(a) Find the set of all possible values of the function  $f$ .

(b) If  $f(10) \neq 0$  and  $f(2) = 0$ , find the set of all integers  $n$  such that  $f(n) \neq 0$ .

**Solution:** Setting  $y = 0$  in the condition (ii), we get

$$(f(x) - f(0))f(x) = 0,$$

for all  $x$  (since  $f(0) \neq 0$ ). Thus either  $f(x) = 0$  or  $f(x) = f(0)$ , for all  $x \in \mathbb{Z}$ . Now taking  $x = y = 0$  in (i), we see that  $f(0) + f(0)^2 = 2f(0)$ . This shows

that  $f(0) = 0$  or  $f(0) = 1$ . Since  $f(0) \neq 0$ , we must have  $f(0) = 1$ . We conclude that

$$\text{either } f(x) = 0 \text{ or } f(x) = 1 \text{ for each } x \in \mathbb{Z}.$$

This shows that the set of all possible value of  $f(x)$  is  $\{0, 1\}$ . This completes (a).

Let  $S = \{n \in \mathbb{Z} \mid f(n) \neq 0\}$ . Hence we must have  $S = \{n \in \mathbb{Z} \mid f(n) = 1\}$  by (a). Since  $f(1) = 0$ , 1 is not in  $S$ . And  $f(0) = 1$  implies that  $0 \in S$ . Take any  $x \in \mathbb{Z}$  and  $y \in S$ . Using (ii), we get

$$f(xy) + f(x) = f(x) + 1.$$

This shows that  $xy \in S$ . If  $x \in \mathbb{Z}$  and  $y \in \mathbb{Z}$  are such that  $xy \in S$ , then (ii) gives

$$1 + f(x)f(y) = f(x) + f(y).$$

Thus  $(f(x) - 1)(f(y) - 1) = 0$ . It follows that  $f(x) = 1$  or  $f(y) = 1$ ; i.e., either  $x \in S$  or  $y \in S$ . We also observe from (ii) that  $x \in S$  and  $y \in S$  implies that  $f(x - y) = 1$  so that  $x - y \in S$ . Thus  $S$  has the properties:

(A)  $x \in \mathbb{Z}$  and  $y \in S$  implies  $xy \in S$ ;

(B)  $x, y \in \mathbb{Z}$  and  $xy \in S$  implies  $x \in S$  or  $y \in S$ ;

(C)  $x, y \in S$  implies  $x - y \in S$ .

Now we know that  $f(10) \neq 0$  and  $f(2) = 0$ . Hence  $f(10) = 1$  and  $10 \in S$ ; and  $2 \notin S$ . Writing  $10 = 2 \times 5$  and using (B), we conclude that  $5 \in S$  and  $f(5) = 1$ . Hence  $f(5k) = 1$  for all  $k \in \mathbb{Z}$  by (A).

Suppose  $f(5k + l) = 1$  for some  $l$ ,  $1 \leq l \leq 4$ . Then  $5k + l \in S$ . Choose  $u \in \mathbb{Z}$  such that  $lu \equiv 1 \pmod{5}$ . We have  $(5k + l)u \in S$  by (A). Moreover,  $lu = 1 + 5m$  for some  $m \in \mathbb{Z}$  and

$$(5k + l)u = 5ku + lu = 5ku + 5m + 1 = 5(ku + m) + 1.$$

This shows that  $5(ku + m) + 1 \in S$ . However, we know that  $5(ku + m) \in S$ . By (C),  $1 \in S$  which is a contradiction. We conclude that  $5k + l \notin S$  for any  $l$ ,  $1 \leq l \leq 4$ . Thus

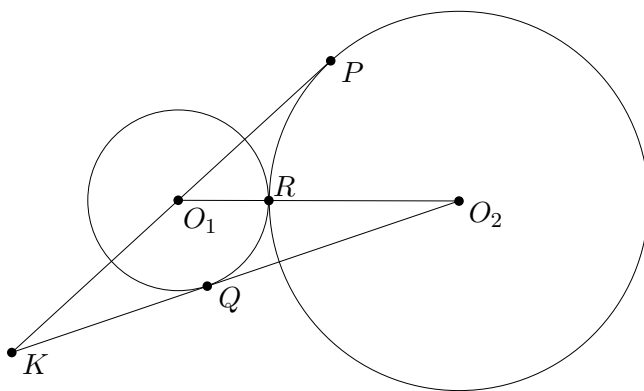
$$S = \{5k \mid k \in \mathbb{Z}\}.$$

—————00000—————

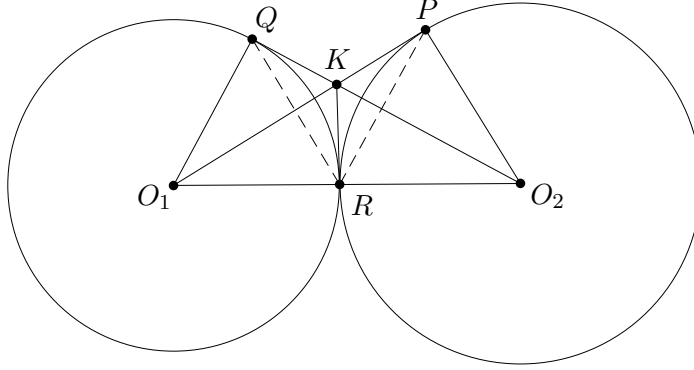
# Problems and solutions: INMO 2013

**Problem 1.** Let  $\Gamma_1$  and  $\Gamma_2$  be two circles touching each other externally at  $R$ . Let  $l_1$  be a line which is tangent to  $\Gamma_2$  at  $P$  and passing through the center  $O_1$  of  $\Gamma_1$ . Similarly, let  $l_2$  be a line which is tangent to  $\Gamma_2$  at  $Q$  and passing through the center  $O_2$  of  $\Gamma_2$ . Suppose  $l_1$  and  $l_2$  are not parallel and intersect at  $K$ . If  $KP = KQ$ , prove that the triangle  $PQR$  is equilateral.

**Solution.** Suppose that  $P$  and  $Q$  lie on the opposite sides of line joining  $O_1$  and  $O_2$ . By symmetry we may assume that the configuration is as shown in the figure below. Then we have  $KP > KO_1 > KQ$  since  $KO_1$  is the hypotenuse of triangle  $KQO_1$ . This is a contradiction to the given assumption, and therefore  $P$  and  $Q$  lie on the same side of the line joining  $O_1$  and  $O_2$ .



Since  $KP = KQ$  it follows that  $K$  lies on the radical axis of the given circles, which is the common tangent at  $R$ . Therefore  $KP = KQ = KR$  and hence  $K$  is the circumcenter of  $\triangle PQR$ .



On the other hand,  $\triangle KQO_1$  and  $\triangle KRO_1$  are both right-angled triangles with  $KQ = KR$  and  $QO_1 = RO_1$ , and hence the two triangles are congruent. Therefore  $\widehat{QKO_1} = \widehat{RKO_1}$ , so  $KO_1$ , and hence  $PK$  is perpendicular to  $QR$ . Similarly,  $QK$  is perpendicular to  $PR$ , so it follows that  $K$  is the orthocenter of  $\triangle PQR$ . Hence we have that  $\triangle PQR$  is equilateral.  $\square$

**Alternate solution.** We again rule out the possibility that  $P$  and  $Q$  are on the opposite side of the line joining  $O_1O_2$ , and assume that they are on the same side.

Observe that  $\triangle KPO_2$  is congruent to  $\triangle KQO_1$  (since  $KP = KQ$ ). Therefore  $O_1P = O_2Q = r$  (say). In  $\triangle O_1O_2Q$ , we have  $\widehat{O_1QO_2} = \pi/2$  and  $R$  is the midpoint of the hypotenuse, so  $RQ = RO_1 = r$ . Therefore  $\triangle O_1RQ$  is equilateral, so  $\widehat{QRO_1} = \pi/3$ . Similarly,  $PR = r$  and  $\widehat{PRO_2} = \pi/3$ , hence  $\widehat{PRQ} = \pi/3$ . Since  $PR = QR$  it follows that  $\triangle PQR$  is equilateral.  $\square$

**Problem 2.** Find all positive integers  $m$ ,  $n$ , and primes  $p \geq 5$  such that

$$m(4m^2 + m + 12) = 3(p^n - 1).$$

**Solution.** Rewriting the given equation we have

$$4m^3 + m^2 + 12m + 3 = 3p^n.$$

The left hand side equals  $(4m + 1)(m^2 + 3)$ .

Suppose that  $(4m + 1, m^2 + 3) = 1$ . Then  $(4m + 1, m^2 + 3) = (3p^n, 1), (3, p^n), (p^n, 3)$  or  $(1, 3p^n)$ , a contradiction since  $4m + 1, m^2 + 3 \geq 4$ . Therefore  $(4m + 1, m^2 + 3) > 1$ .

Since  $4m + 1$  is odd we have  $(4m + 1, m^2 + 3) = (4m + 1, 16m^2 + 48) = (4m + 1, 49) = 7$  or  $49$ . This proves that  $p = 7$ , and  $4m + 1 = 3 \cdot 7^k$  or  $7^k$  for some natural number  $k$ . If  $(4m + 1, 49) = 7$  then we have  $k = 1$  and  $4m + 1 = 21$  which does not lead to a solution. Therefore  $(4m + 1, m^2 + 3) = 49$ . If  $7^3$  divides  $4m + 1$  then it does not divide  $m^2 + 3$ , so we get  $m^2 + 3 \leq 3 \cdot 7^2 < 7^3 \leq 4m + 1$ . This implies  $(m - 2)^2 < 2$ , so  $m \leq 3$ , which does not lead to a solution. Therefore we have  $4m + 1 = 49$  which implies  $m = 12$  and  $n = 4$ . Thus  $(m, n, p) = (12, 4, 7)$  is the only solution.  $\square$

**Problem 3.** Let  $a, b, c, d$  be positive integers such that  $a \geq b \geq c \geq d$ . Prove that the equation  $x^4 - ax^3 - bx^2 - cx - d = 0$  has no integer solution.

**Solution.** Suppose that  $m$  is an integer root of  $x^4 - ax^3 - bx^2 - cx - d = 0$ . As  $d \neq 0$ , we have  $m \neq 0$ . Suppose now that  $m > 0$ . Then  $m^4 - am^3 = bm^2 + cm + d > 0$  and hence  $m > a \geq d$ . On the other hand  $d = m(m^3 - am^2 - bm - c)$  and hence  $m$  divides  $d$ , so  $m \leq d$ , a contradiction. If  $m < 0$ , then writing  $n = -m > 0$  we have  $n^4 + an^3 - bn^2 + cn - d = n^4 + n^2(an - b) + (cn - d) > 0$ , a contradiction. This proves that the given polynomial has no integer roots.  $\square$

**Problem 4.** Let  $n$  be a positive integer. Call a nonempty subset  $S$  of  $\{1, 2, \dots, n\}$  good if the arithmetic mean of the elements of  $S$  is also an integer. Further let  $t_n$  denote the number of good subsets of  $\{1, 2, \dots, n\}$ . Prove that  $t_n$  and  $n$  are both odd or both even.

**Solution.** We show that  $T_n - n$  is even. Note that the subsets  $\{1\}, \{2\}, \dots, \{n\}$  are good. Among the other good subsets, let  $A$  be the collection of subsets with an integer average which belongs to the subset, and let  $B$  be the collection of subsets with an integer average which is not a member of the subset. Then there is a bijection between  $A$  and  $B$ , because removing the average takes a member of  $A$  to a member of  $B$ ; and including the average in a member of  $B$  takes it to its inverse. So  $T_n - n = |A| + |B|$  is even.  $\square$

**Alternate solution.** Let  $S = \{1, 2, \dots, n\}$ . For a subset  $A$  of  $S$ , let  $\bar{A} = \{n + 1 - a | a \in A\}$ . We call a subset  $A$  symmetric if  $\bar{A} = A$ . Note that the arithmetic mean of a symmetric subset is  $(n + 1)/2$ . Therefore, if  $n$  is even, then there are no symmetric good subsets, while if  $n$  is odd then every symmetric subset is good.

If  $A$  is a proper good subset of  $S$ , then so is  $\bar{A}$ . Therefore, all the good subsets that are not symmetric can be paired. If  $n$  is even then this proves that  $t_n$  is even. If  $n$  is odd, we have to show that there are odd number of symmetric subsets. For this, we note that a symmetric subset contains the element  $(n + 1)/2$  if and only if it has odd number of elements. Therefore, for any natural number  $k$ , the number of symmetric subsets of size  $2k$  equals the number of symmetric subsets of size  $2k + 1$ . The result now follows since there is exactly one symmetric subset with only one element.  $\square$

**Problem 5.** In an acute triangle  $ABC$ ,  $O$  is the circumcenter,  $H$  is the orthocenter and  $G$  is the centroid. Let  $OD$  be perpendicular to  $BC$  and  $HE$  be perpendicular to  $CA$ , with  $D$  on  $BC$  and  $E$  on  $CA$ . Let  $F$  be the midpoint of  $AB$ . Suppose the areas of triangles  $ODC$ ,  $HEA$  and  $GFB$  are equal. Find all the possible values of  $\hat{C}$ .

**Solution.** Let  $R$  be the circumradius of  $\triangle ABC$  and  $\Delta$  its area. We have  $OD = R \cos A$  and  $DC = \frac{a}{2}$ , so

$$[ODC] = \frac{1}{2} \cdot OD \cdot DC = \frac{1}{2} \cdot R \cos A \cdot R \sin A = \frac{1}{2} R^2 \sin A \cos A. \quad (1)$$

Again  $HE = 2R \cos C \cos A$  and  $EA = c \cos A$ . Hence

$$[HEA] = \frac{1}{2} \cdot HE \cdot EA = \frac{1}{2} \cdot 2R \cos C \cos A \cdot c \cos A = 2R^2 \sin C \cos C \cos^2 A. \quad (2)$$

Further

$$[GFB] = \frac{\Delta}{6} = \frac{1}{6} \cdot 2R^2 \sin A \sin B \sin C = \frac{1}{3} R^2 \sin A \sin B \sin C. \quad (3)$$

Equating (1) and (2) we get  $\tan A = 4 \sin C \cos C$ . And equating (1) and (3), and using this relation we get

$$\begin{aligned} 3 \cos A &= 2 \sin B \sin C = 2 \sin(C + A) \sin C \\ &= 2(\sin C + \cos C \tan A) \sin C \cos A \\ &= 2 \sin^2 C (1 + 4 \cos^2 C) \cos A. \end{aligned}$$

Since  $\cos A \neq 0$  we get  $3 = 2t(-4t + 5)$  where  $t = \sin^2 C$ . This implies  $(4t - 3)(2t - 1) = 0$  and therefore, since  $\sin C > 0$ , we get  $\sin C = \sqrt{3}/2$  or  $\sin C = 1/\sqrt{2}$ . Because  $\triangle ABC$  is acute, it follows that  $\hat{C} = \pi/3$  or  $\pi/4$ .

We observe that the given conditions are satisfied in an equilateral triangle, so  $\hat{C} = \pi/3$  is a possibility. Also, the conditions are satisfied in a triangle where  $\hat{C} = \pi/4$ ,  $\hat{A} = \tan^{-1} 2$  and  $\hat{B} = \tan^{-1} 3$ . Therefore  $\hat{C} = \pi/4$  is also a possibility.

Thus the two possible values of  $\hat{C}$  are  $\pi/3$  and  $\pi/4$ . □

**Problem 6.** Let  $a, b, c, x, y, z$  be positive real numbers such that  $a + b + c = x + y + z$  and  $abc = xyz$ . Further, suppose that  $a \leq x < y < z \leq c$  and  $a < b < c$ . Prove that  $a = x, b = y$  and  $c = z$ .

**Solution.** Let

$$f(t) = (t - x)(t - y)(t - z) - (t - a)(t - b)(t - c).$$

Then  $f(t) = kt$  for some constant  $k$ . Note that  $ka = f(a) = (a - x)(a - y)(a - z) \leq 0$  and hence  $k \leq 0$ . Similarly,  $kc = f(c) = (c - x)(c - y)(c - z) \geq 0$  and hence  $k \geq 0$ . Combining the two, it follows that  $k = 0$  and that  $f(a) = f(c) = 0$ . These equalities imply that  $a = x$  and  $c = z$ , and then it also follows that  $b = y$ . □

# 29<sup>th</sup> Indian National Mathematical Olympiad-2014

February 02, 2014

1. In a triangle  $ABC$ , let  $D$  be a point on the segment  $BC$  such that  $AB + BD = AC + CD$ . Suppose that the points  $B, C$  and the centroids of triangles  $ABD$  and  $ACD$  lie on a circle. Prove that  $AB = AC$ .

**Solution.** Let  $G_1, G_2$  denote the centroids of triangles  $ABD$  and  $ACD$ . Then  $G_1, G_2$  lie on the line parallel to  $BC$  that passes through the centroid of triangle  $ABC$ . Therefore  $BG_1G_2C$  is an isosceles trapezoid. Therefore it follows that  $BG_1 = CG_2$ . This proves that  $AB^2 + BD^2 = AC^2 + CD^2$ . Hence it follows that  $AB \cdot BD = AC \cdot CD$ . Therefore the sets  $\{AB, BD\}$  and  $\{AC, CD\}$  are the same (since they are both equal to the set of roots of the same polynomial). Note that if  $AB = CD$  then  $AC = BD$  and then  $AB + AC = BC$ , a contradiction. Therefore it follows that  $AB = AC$ .

2. Let  $n$  be a natural number. Prove that

$$\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor + \lfloor \sqrt{n} \rfloor$$

is **even**. (Here  $\lfloor x \rfloor$  denotes the largest integer smaller than or equal to  $x$ .)

**Solution.** Let  $f(n)$  denote the given equation. Then  $f(1) = 2$  which is even. Now suppose that  $f(n)$  is even for some  $n \geq 1$ . Then

$$\begin{aligned} f(n+1) &= \left\lfloor \frac{n+1}{1} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \cdots + \left\lfloor \frac{n+1}{n+1} \right\rfloor + \lfloor \sqrt{n+1} \rfloor \\ &= \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor + \lfloor \sqrt{n+1} \rfloor + \sigma(n+1), \end{aligned}$$

where  $\sigma(n+1)$  denotes the number of positive divisors of  $n+1$ . This follows from  $\left\lfloor \frac{n+1}{k} \right\rfloor = \left\lfloor \frac{n}{k} \right\rfloor + 1$  if  $k$  divides  $n+1$ , and  $\left\lfloor \frac{n+1}{k} \right\rfloor = \left\lfloor \frac{n}{k} \right\rfloor$  otherwise. Note that  $\lfloor \sqrt{n+1} \rfloor = \lfloor \sqrt{n} \rfloor$  unless  $n+1$  is a square, in which case  $\lfloor \sqrt{n+1} \rfloor = \lfloor \sqrt{n} \rfloor + 1$ . On the other hand  $\sigma(n+1)$  is odd if and only if  $n+1$  is a square. Therefore it follows that  $f(n+1) = f(n) + 2l$  for some integer  $l$ . This proves that  $f(n+1)$  is even.

Thus it follows by induction that  $f(n)$  is even for all natural number  $n$ .

3. Let  $a, b$  be natural numbers with  $ab > 2$ . Suppose that the sum of their greatest common divisor and least common multiple is divisible by  $a+b$ . Prove that the quotient is at most  $(a+b)/4$ . When is this quotient exactly equal to  $(a+b)/4$ ?

**Solution.** Let  $g$  and  $l$  denote the greatest common divisor and the least common multiple, respectively, of  $a$  and  $b$ . Then  $gl = ab$ . Therefore  $g + l \leq ab + 1$ . Suppose that  $(g + l)/(a + b) > (a + b)/4$ . Then we have  $ab + 1 > (a + b)^2/4$ , so we get  $(a - b)^2 < 4$ . Assuming,  $a \geq b$  we either have  $a = b$  or  $a = b + 1$ . In the former case,  $g = l = a$  and the quotient is  $(g + l)/(a + b) = 1 \leq (a + b)/4$ . In the latter case,  $g = 1$  and  $l = b(b + 1)$  so we get that  $2b + 1$  divides  $b^2 + b + 1$ . Therefore  $2b + 1$  divides  $4(b^2 + b + 1) - (2b + 1)^2 = 3$  which implies that  $b = 1$  and  $a = 2$ , a contradiction to the given assumption that  $ab > 2$ . This shows that  $(g + l)/(a + b) \leq (a + b)/4$ . Note that for the equality to hold, we need that either  $a = b = 2$  or,  $(a - b)^2 = 4$  and  $g = 1, l = ab$ . The latter case happens if and only if  $a$  and  $b$  are two consecutive odd numbers. (If  $a = 2k + 1$  and  $b = 2k - 1$  then  $a + b = 4k$  divides  $ab + 1 = 4k^2$  and the quotient is precisely  $(a + b)/4$ .)

4. Written on a blackboard is the polynomial  $x^2 + x + 2014$ . Calvin and Hobbes take turns alternatively (starting with Calvin) in the following game. During his turn, Calvin should either increase or decrease the coefficient of  $x$  by 1. And during his turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.

**Solution.** For  $i \geq 0$ , let  $f_i(x)$  denote the polynomial on the blackboard after Hobbes'  $i$ -th turn. We let Calvin decrease the coefficient of  $x$  by 1. Therefore  $f_{i+1}(2) = f_i(2) - 1$  or  $f_{i+1}(2) = f_i(2) - 3$  (depending on whether Hobbes increases or decreases the constant term). So for some  $i$ , we have  $0 \leq f_i(2) \leq 2$ . If  $f_i(2) = 0$  then Calvin has won the game. If  $f_i(2) = 2$  then Calvin wins the game by reducing the coefficient of  $x$  by 1. If  $f_i(2) = 1$  then  $f_{i+1}(2) = 0$  or  $f_{i+1}(2) = -2$ . In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of  $x$  by 1.

5. In an acute-angled triangle  $ABC$ , a point  $D$  lies on the segment  $BC$ . Let  $O_1, O_2$  denote the circumcentres of triangles  $ABD$  and  $ACD$ , respectively. Prove that the line joining the circumcentre of triangle  $ABC$  and the orthocentre of triangle  $O_1O_2D$  is parallel to  $BC$ .

**Solution.** Without loss of generality assume that  $\angle ADC \geq 90^\circ$ . Let  $O$  denote the circumcenter of triangle  $ABC$  and  $K$  the orthocentre of triangle  $O_1O_2D$ . We shall first show that the points  $O$  and  $K$  lie on the circumcircle of triangle  $AO_1O_2$ . Note that circumcircles of triangles  $ABD$  and  $ACD$  pass through the points  $A$  and  $D$ , so  $AD$  is perpendicular to  $O_1O_2$  and, triangle  $AO_1O_2$  is congruent to triangle  $DO_1O_2$ . In particular,  $\angle AO_1O_2 = \angle O_2O_1D = \angle B$  since  $O_2O_1$  is the perpendicular bisector of  $AD$ . On the other hand since  $OO_2$  is the perpendicular bisector of  $AC$  it follows that  $\angle AOO_2 = \angle B$ . This shows that  $O$  lies on the circumcircle of triangle  $AO_1O_2$ . Note also that, since  $AD$  is perpendicular to  $O_1O_2$ , we have  $\angle O_2KA = 90^\circ - \angle O_1O_2K = \angle O_2O_1D = \angle B$ . This proves that  $K$  also lies on the circumcircle of triangle  $AO_1O_2$ .

Therefore  $\angle AKO = 180^\circ - \angle AO_2O = \angle ADC$  and hence  $OK$  is parallel to  $BC$ .

**Remark.** The result is true even for an obtuse-angled triangle.

6. Let  $n$  be a natural number and  $X = \{1, 2, \dots, n\}$ . For subsets  $A$  and  $B$  of  $X$  we define  $A\Delta B$  to be the set of all those elements of  $X$  which belong to exactly one of  $A$  and  $B$ . Let  $\mathcal{F}$  be a collection of subsets of  $X$  such that for any two distinct elements  $A$  and  $B$  in  $\mathcal{F}$  the set  $A\Delta B$  has at least two elements. Show that  $\mathcal{F}$  has at most  $2^{n-1}$  elements. Find all such collections  $\mathcal{F}$  with  $2^{n-1}$  elements.

**Solution.** For each subset  $A$  of  $\{1, 2, \dots, n-1\}$ , we pair it with  $A \cup \{n\}$ . Note that for any such pair  $(A, B)$  not both  $A$  and  $B$  can be in  $\mathcal{F}$ . Since there are  $2^{n-1}$  such pairs it follows that  $\mathcal{F}$  can have at most  $2^{n-1}$  elements.

We shall show by induction on  $n$  that if  $|\mathcal{F}| = 2^{n-1}$  then  $\mathcal{F}$  contains either all the subsets with odd number of elements or all the subsets with even number of elements. The result is easy to see for  $n = 1$ . Suppose that the result is true for  $n = m - 1$ . We now consider the case  $n = m$ . Let  $\mathcal{F}_1$  be the set of those elements in  $\mathcal{F}$  which contain  $m$  and  $\mathcal{F}_2$  be the set of those elements which do not contain  $m$ . By induction,  $\mathcal{F}_2$  can have at most  $2^{m-2}$  elements. Further, for each element  $A$  of  $\mathcal{F}_1$  we consider  $A \setminus \{m\}$ . This new collection also satisfies the required property, so it follows that  $\mathcal{F}_1$  has at most  $2^{m-2}$  elements. Thus, if  $|\mathcal{F}| = 2^{m-1}$  then it follows that  $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{m-2}$ . Further, by induction hypothesis,  $\mathcal{F}_2$  contains all those subsets of  $\{1, 2, \dots, m-1\}$  with (say) even number of elements. It then follows that  $\mathcal{F}_1$  contains all those subsets of  $\{1, 2, \dots, m\}$  which contain the element  $m$  and which contains an even number of elements. This proves that  $\mathcal{F}$  contains either all the subsets with odd number of elements or all the subsets with even number of elements.

———— ★ ★ ★ ★ ★ ————

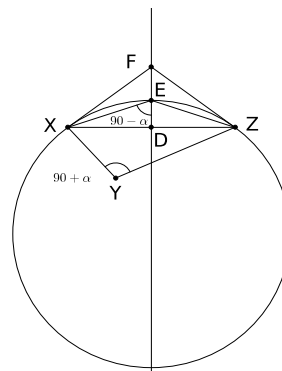
## Problems and Solutions: INMO-2015

1. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $BD$  be the altitude from  $B$  on to  $AC$ . Let  $P$ ,  $Q$  and  $I$  be the incentres of triangles  $ABD$ ,  $CBD$  and  $ABC$  respectively. Show that the circumcentre of the triangle  $PIQ$  lies on the hypotenuse  $AC$ .

**Solution:** We begin with the following lemma:

**Lemma:** Let  $XYZ$  be a triangle with  $\angle XYZ = 90 + \alpha$ . Construct an isosceles triangle  $XEZ$ , externally on the side  $XZ$ , with base angle  $\alpha$ . Then  $E$  is the circumcentre of  $\triangle XYZ$ .

**Proof of the Lemma:** Draw  $ED \perp XZ$ . Then  $DE$  is the perpendicular bisector of  $XZ$ . We also observe that  $\angle XED = \angle ZED = 90 - \alpha$ . Observe that  $E$  is on the perpendicular bisector of  $XZ$ . Construct the circumcircle of  $XYZ$ . Draw perpendicular bisector of  $XY$  and let it meet  $DE$  in  $F$ . Then  $F$  is the circumcentre of  $\triangle XYZ$ . Join  $XF$ . Then  $\angle XFD = 90 - \alpha$ . But we know that  $\angle XED = 90 - \alpha$ . Hence  $E = F$ .



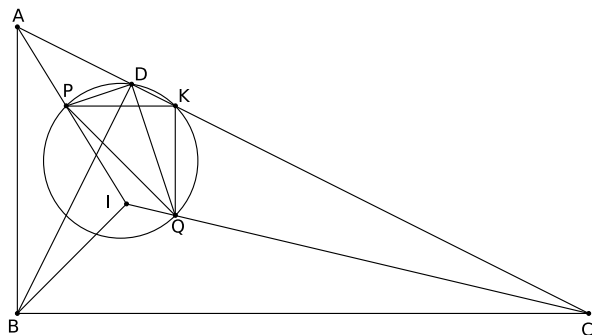
Let  $r_1$ ,  $r_2$  and  $r$  be the inradii of the triangles  $ABD$ ,  $CBD$  and  $ABC$  respectively. Join  $PD$  and  $DQ$ . Observe that  $\angle PDQ = 90^\circ$ . Hence

$$PQ^2 = PD^2 + DQ^2 = 2r_1^2 + 2r_2^2.$$

Let  $s_1 = (AB + BD + DA)/2$ . Observe that  $BD = ca/b$  and  $AD = \sqrt{AB^2 - BD^2} = \sqrt{c^2 - (ca/b)^2} = c^2/b$ . This gives  $s_1 = cs/b$ . But  $r_1 = s_1 - c = (c/b)(s - b) = cr/b$ . Similarly,  $r_2 = ar/b$ . Hence

$$PQ^2 = 2r^2 \left( \frac{c^2 + a^2}{b^2} \right) = 2r^2.$$

Consider  $\triangle PIQ$ . Observe that  $\angle PIQ = 90 + (B/2) = 135$ . Hence  $PQ$  subtends  $90^\circ$  on the circumference of the circumcircle of  $\triangle PIQ$ . But we have seen that  $\angle PDQ = 90^\circ$ . Now construct a circle with  $PQ$  as diameter. Let it cut  $AC$  again in  $K$ . It follows that  $\angle PKQ = 90^\circ$  and the points  $P, D, K, Q$  are concyclic. We also notice  $\angle KPQ = \angle KDQ = 45^\circ$  and  $\angle PQK = \angle PDK = 45^\circ$ .

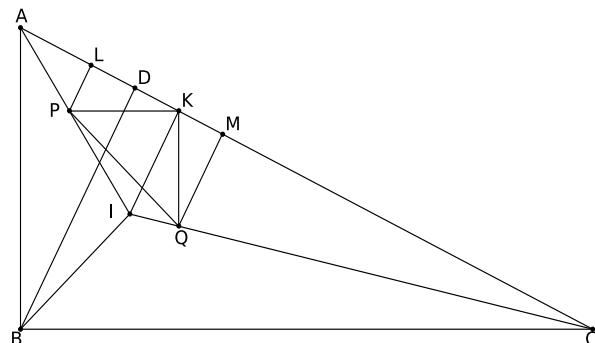


Thus  $PKQ$  is an isosceles right-angled triangle with  $KP = KQ$ . Therefore  $KP^2 + KQ^2 = PQ^2 = 2r^2$  and hence  $KP = KQ = r$ .

Now  $\angle PIQ = 90 + 45$  and  $\angle PKQ = 2 \times 45^\circ = 90^\circ$  with  $KP = KQ = r$ . Hence  $K$  is the circumcentre of  $\triangle PIQ$ .

(Incidentally, This also shows that  $KI = r$  and hence  $K$  is the point of contact of the incircle of  $\triangle ABC$  with  $AC$ .)

**Solution 2:** Here we use computation to prove that the point of contact  $K$  of the incircle with  $AC$  is the circumcentre of  $\triangle PIQ$ . We show that  $KP = KQ = r$ . Let  $r_1$  and  $r_2$  be the inradii of triangles  $ABD$  and  $CBD$  respectively. Draw  $PL \perp AC$  and  $QM \perp AC$ . If  $s_1$  is the semiperimeter of  $\triangle ABD$ , then  $AL = s_1 - BD$ .



But

$$s_1 = \frac{AB + BD + DA}{2}, \quad BD = \frac{ca}{b}, \quad AD = \frac{c^2}{b}$$

Hence  $s_1 = cs/b$ . This gives  $r_1 = s_1 - c = cr/b$ ,  $AL = s_1 - BD = c(s-a)/b$ .

Hence  $KL = AK - AL = (s-a) - \frac{c(s-a)}{b} = \frac{(b-c)(s-a)}{b}$ . We observe that

$$2r^2 = \frac{(c+a-b)^2}{2} = \frac{c^2 + a^2 + b^2 - 2bc - 2ab + 2ca}{2} = (b^2 - ba - bc + ac) = (b-c)(b-a).$$

This gives

$$\begin{aligned} (s-a)(b-c) &= (s-b+b-a)(b-c) = r(b-c) + (b-a)(b-c) \\ &= r(b-c) + 2r^2 = r(b-c+c+a-b) = ra. \end{aligned}$$

Thus  $KL = ra/b$ . Finally,

$$KP^2 = KL^2 + LP^2 = \frac{r^2 a^2}{b^2} + \frac{r^2 + c^2}{b^2} = r^2.$$

Thus  $KP = r$ . Similarly,  $KQ = r$ . This gives  $KP = KI = KQ = r$  and therefore  $K$  is the circumcentre of  $\triangle KIQ$ .

(Incidentally, this also shows that  $KL = ca/b = r_2$  and  $KM = r_1$ .)

- For any natural number  $n > 1$ , write the infinite decimal expansion of  $1/n$  (for example, we write  $1/2 = 0.4\bar{9}$  as its infinite decimal expansion, not 0.5). Determine the length of the non-periodic part of the (infinite) decimal expansion of  $1/n$ .

**Solution:** For any prime  $p$ , let  $\nu_p(n)$  be the maximum power of  $p$  dividing  $n$ ; ie  $p^{\nu_p(n)}$  divides  $n$  but not higher power. Let  $r$  be the

length of the non-periodic part of the infinite decimal expansion of  $1/n$ .

Write

$$\frac{1}{n} = 0.a_1a_2 \cdots a_r \overline{b_1b_2 \cdots b_s}.$$

We show that  $r = \max(\nu_2(n), \nu_5(n))$ .

Let  $a$  and  $b$  be the numbers  $a_1a_2 \cdots a_r$  and  $b = b_1b_2 \cdots b_s$  respectively. (Here  $a_1$  and  $b_1$  can be both 0.) Then

$$\frac{1}{n} = \frac{1}{10^r} \left( a + \sum_{k \geq 1} \frac{b}{(10^s)^k} \right) = \frac{1}{10^r} \left( a + \frac{b}{10^s - 1} \right).$$

Thus we get  $10^r(10^s - 1) = n((10^s - 1)a + b)$ . It shows that  $r \geq \max(\nu_2(n), \nu_5(n))$ . Suppose  $r > \max(\nu_2(n), \nu_5(n))$ . Then 10 divides  $b - a$ . Hence the last digits of  $a$  and  $b$  are equal:  $a_r = b_s$ . This means

$$\frac{1}{n} = 0.a_1a_2 \cdots a_{r-1} \overline{b_sb_1b_2 \cdots b_{s-1}}.$$

This contradicts the definition of  $r$ . Therefore  $r = \max(\nu_2(n), \nu_5(n))$ .

3. Find all real functions  $f$  from  $\mathbb{R} \rightarrow \mathbb{R}$  satisfying the relation

$$f(x^2 + yf(x)) = xf(x + y).$$

**Solution:** Put  $x = 0$  and we get  $f(yf(0)) = 0$ . If  $f(0) \neq 0$ , then  $yf(0)$  takes all real values when  $y$  varies over real line. We get  $f(x) \equiv 0$ . Suppose  $f(0) = 0$ . Taking  $y = -x$ , we get  $f(x^2 - xf(x)) = 0$  for all real  $x$ .

Suppose there exists  $x_0 \neq 0$  in  $\mathbb{R}$  such that  $f(x_0) = 0$ . Putting  $x = x_0$  in the given relation we get

$$f(x_0^2) = x_0f(x_0 + y),$$

for all  $y \in \mathbb{R}$ . Now the left side is a constant and hence it follows that  $f$  is a constant function. But the only constant function which satisfies the equation is identically zero function, which is already obtained. Hence we may consider the case where  $f(x) \neq 0$  for all  $x \neq 0$ .

Since  $f(x^2 - xf(x)) = 0$ , we conclude that  $x^2 - xf(x) = 0$  for all  $x \neq 0$ . This implies that  $f(x) = x$  for all  $x \neq 0$ . Since  $f(0) = 0$ , we conclude that  $f(x) = x$  for all  $x \in \mathbb{R}$ .

Thus we have two functions:  $f(x) \equiv 0$  and  $f(x) = x$  for all  $x \in \mathbb{R}$ .

4. There are four basket-ball players  $A, B, C, D$ . Initially, the ball is with  $A$ . The ball is always passed from one person to a different person. In how many ways can the ball come back to  $A$  after **seven** passes? (For example  $A \rightarrow C \rightarrow B \rightarrow D \rightarrow A \rightarrow B \rightarrow C \rightarrow A$  and

$A \rightarrow D \rightarrow A \rightarrow D \rightarrow C \rightarrow A \rightarrow B \rightarrow A$  are two ways in which the ball can come back to  $A$  after seven passes.)

**Solution:** Let  $x_n$  be the number of ways in which  $A$  can get back the ball after  $n$  passes. Let  $y_n$  be the number of ways in which the ball goes back to a fixed person other than  $A$  after  $n$  passes. Then

$$x_n = 3y_{n-1},$$

and

$$y_n = x_{n-1} + 2y_{n-1}.$$

We also have  $x_1 = 0$ ,  $x_2 = 3$ ,  $y_1 = 1$  and  $y_2 = 2$ .

Eliminating  $y_n$  and  $y_{n-1}$ , we get  $x_{n+1} = 3x_{n-1} + 2x_n$ . Thus

$$\begin{aligned} x_3 &= 3x_1 + 2x_2 = 2 \times 3 = 6; \\ x_4 &= 3x_2 + 2x_3 = (3 \times 3) + (2 \times 6) = 9 + 12 = 21; \\ x_5 &= 3x_3 + 2x_4 = (3 \times 6) + (2 \times 21) = 18 + 42 = 60; \\ x_6 &= 3x_4 + 2x_5 = (3 \times 21) + (2 \times 60) = 63 + 120 = 183; \\ x_7 &= 3x_5 + 2x_6 = (3 \times 60) + (2 \times 183) = 180 + 366 = 546. \end{aligned}$$

**Alternate solution:** Since the ball goes back to one of the other 3 persons, we have

$$x_n + 3y_n = 3^n,$$

since there are  $3^n$  ways of passing the ball in  $n$  passes. Using  $x_n = 3y_{n-1}$ , we obtain

$$x_{n-1} + x_n = 3^{n-1},$$

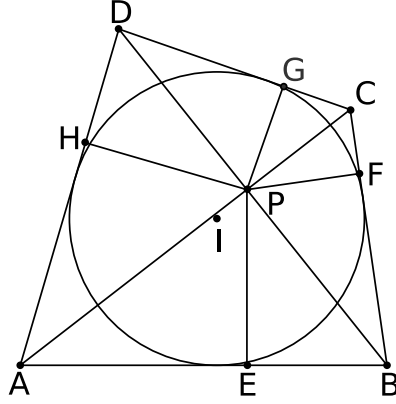
with  $x_1 = 0$ . Thus

$$\begin{aligned} x_7 &= 3^6 - x_6 = 3^6 - 3^5 + x_5 = 3^6 - 3^5 + 3^4 - x_4 = 3^6 - 3^5 + 3^4 - 3^3 + x_3 \\ &= 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - x_2 = 3^6 - 3^5 + 3^4 - 3^3 + 3^2 - 3 \\ &= (2 \times 3^5) + (2 \times 3^3) + (2 \times 3) = 486 + 54 + 6 = 546. \end{aligned}$$

5. Let  $ABCD$  be a convex quadrilateral. Let the diagonals  $AC$  and  $BD$  intersect in  $P$ . Let  $PE$ ,  $PF$ ,  $PG$  and  $PH$  be the altitudes from  $P$  on to the sides  $AB$ ,  $BC$ ,  $CD$  and  $DA$  respectively. Show that  $ABCD$  has an incircle if and only if

$$\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}.$$

**Solution:** Let  $AP = p$ ,  $BP = q$ ,  $CP = r$ ,  $DP = s$ ;  $AB = a$ ,  $BC = b$ ,  $CD = c$  and  $DA = d$ . Let  $\angle APB = \angle CPD = \theta$ . Then  $\angle BPC = \angle DPA = \pi - \theta$ . Let us also write  $PE = h_1$ ,  $PF = h_2$ ,  $PG = h_3$  and  $PH = h_4$ .



Observe that

$$h_1 a = pq \sin \theta, \quad h_2 b = qr \sin \theta, \quad h_3 c = rs \sin \theta, \quad h_4 d = sp \sin \theta.$$

Hence

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

is equivalent to

$$\frac{a}{pq} + \frac{c}{rs} = \frac{b}{qr} + \frac{d}{sp}.$$

This is the same as

$$ars + cpq = bsp + dqr.$$

Thus we have to prove that  $a + c = b + d$  if and only if  $ars + cpq = bsp + dqr$ .

Now we can write  $a + c = b + d$  as

$$a^2 + c^2 + 2ac = b^2 + d^2 + 2bd.$$

But we know that

$$\begin{aligned} a^2 &= p^2 + q^2 - 2pq \cos \theta, & c^2 &= r^2 + s^2 - 2rs \cos \theta \\ b^2 &= q^2 + r^2 + 2qr \cos \theta, & d^2 &= p^2 + s^2 + 2ps \cos \theta, \end{aligned}$$

Hence  $a + c = b + d$  is equivalent to

$$-pq \cos \theta + -rs \cos \theta + ac = ps \cos \theta + qr \cos \theta + bd.$$

Similarly, by squaring  $ars + cpq = bsp + dqr$  we can show that it is equivalent to

$$-pq \cos \theta + -rs \cos \theta + ac = ps \cos \theta + qr \cos \theta + bd.$$

We conclude that  $a + c = b + d$  is equivalent to  $cpq + ars = bps + dqr$ .

Hence  $ABCD$  has an in circle if and only if

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

6. From a set of 11 square integers, show that one can choose 6 numbers  $a^2, b^2, c^2, d^2, e^2, f^2$  such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$

**Solution:** The first observation is that we can find 5 pairs of squares such that the two numbers in a pair have the same parity. We can see this as follows:

Odd numbers	Even numbers	Odd pairs	Even pairs	Total pairs
0	11	0	5	5
1	10	0	5	5
2	9	1	4	5
3	8	1	4	5
4	7	2	3	5
5	6	2	3	5
6	5	3	2	5
7	4	3	2	5
8	3	4	1	5
9	2	4	1	5
10	1	5	0	5
11	0	5	0	5

Let us take such 5 pairs: say  $(x_1^2, y_1^2), (x_2^2, y_2^2), \dots, (x_5^2, y_5^2)$ . Then  $x_j^2 - y_j^2$  is divisible by 4 for  $1 \leq j \leq 5$ . Let  $r_j$  be the remainder when  $x_j^2 - y_j^2$  is divisible by 3,  $1 \leq j \leq 5$ . We have 5 remainders  $r_1, r_2, r_3, r_4, r_5$ . But these can be 0, 1 or 2. Hence either one of the remainders occur 3 times or each of the remainders occur once. If, for example  $r_1 = r_2 = r_3$ , then 3 divides  $r_1 + r_2 + r_3$ ; if  $r_1 = 0, r_2 = 1$  and  $r_3 = 2$ , then again 3 divides  $r_1 + r_2 + r_3$ . Thus we can always find three remainders whose sum is divisible by 3. This means we can find 3 pairs, say,  $(x_1^2, y_1^2), (x_2^2, y_2^2), (x_3^2, y_3^2)$  such that 3 divides  $(x_1^2 - y_1^2) + (x_2^2 - y_2^2) + (x_3^2 - y_3^2)$ . Since each difference is divisible by 4, we conclude that we can find 6 numbers  $a^2, b^2, c^2, d^2, e^2, f^2$  such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}.$$

# 31<sup>st</sup> Indian National Mathematical Olympiad-2016

Time: 4 hours

January 17, 2016

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions. Maximum marks: 100.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let  $ABC$  be triangle in which  $AB = AC$ . Suppose the orthocentre of the triangle lies on the incircle. Find the ratio  $AB/BC$ .
2. For positive real numbers  $a, b, c$ , which of the following statements necessarily implies  $a = b = c$ : (I)  $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$ , (II)  $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$  ? Justify your answer.
3. Let  $\mathbb{N}$  denote the set of all natural numbers. Define a function  $T : \mathbb{N} \rightarrow \mathbb{N}$  by  $T(2k) = k$  and  $T(2k + 1) = 2k + 2$ . We write  $T^2(n) = T(T(n))$  and in general  $T^k(n) = T^{k-1}(T(n))$  for any  $k > 1$ .
  - (i) Show that for each  $n \in \mathbb{N}$ , there exists  $k$  such that  $T^k(n) = 1$ .
  - (ii) For  $k \in \mathbb{N}$ , let  $c_k$  denote the number of elements in the set  $\{n : T^k(n) = 1\}$ . Prove that  $c_{k+2} = c_{k+1} + c_k$ , for  $k \geq 1$ .
4. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number  $n \geq 3$ , prove that there is a regular  $n$ -sided polygon all of whose vertices are blue.
5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $D$  be a point on  $AC$  such that the inradii of the triangles  $ABD$  and  $CBD$  are equal. If this common value is  $r'$  and if  $r$  is the inradius of triangle  $ABC$ , prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

6. Consider a nonconstant arithmetic progression  $a_1, a_2, \dots, a_n, \dots$ . Suppose there exist relatively prime positive integers  $p > 1$  and  $q > 1$  such that  $a_1^2, a_{p+1}^2$  and  $a_{q+1}^2$  are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

## INMO-2016 problems and solutions

1. Let  $ABC$  be triangle in which  $AB = AC$ . Suppose the orthocentre of the triangle lies on the in-circle. Find the ratio  $AB/BC$ .

**Solution:** Since the triangle is isosceles, the orthocentre lies on the perpendicular  $AD$  from  $A$  on to  $BC$ . Let it cut the in-circle at  $H$ . Now we are given that  $H$  is the orthocentre of the triangle. Let  $AB = AC = b$  and  $BC = 2a$ . Then  $BD = a$ . Observe that  $b > a$  since  $b$  is the hypotenuse and  $a$  is a leg of a right-angled triangle. Let  $BH$  meet  $AC$  in  $E$  and  $CH$  meet  $AB$  in  $F$ . By Pythagoras theorem applied to  $\triangle BDH$ , we get

$$BH^2 = HD^2 + BD^2 = 4r^2 + a^2,$$

where  $r$  is the in-radius of  $ABC$ . We want to compute  $BH$  in another way. Since  $A, F, H, E$  are con-cyclic, we have

$$BH \cdot BE = BF \cdot BA.$$

But  $BF \cdot BA = BD \cdot BC = 2a^2$ , since  $A, F, D, C$  are con-cyclic. Hence  $BH^2 = 4a^4/BE^2$ . But

$$BE^2 = 4a^2 - CE^2 = 4a^2 - BF^2 = 4a^2 - \left(\frac{2a^2}{b}\right)^2 = \frac{4a^2(b^2 - a^2)}{b^2}.$$

This leads to

$$BH^2 = \frac{a^2 b^2}{b^2 - a^2}.$$

Thus we get

$$\frac{a^2 b^2}{b^2 - a^2} = a^2 + 4r^2.$$

This simplifies to  $(a^4/(b^2 - a^2)) = 4r^2$ . Now we relate  $a, b, r$  in another way using area. We know that  $[ABC] = rs$ , where  $s$  is the semi-perimeter of  $ABC$ . We have  $s = (b + b + 2a)/2 = b + a$ . On the other hand area can be calculated using Heron's formula::

$$[ABC]^2 = s(s - 2a)(s - b)(s - b) = (b + a)(b - a)a^2 = a^2(b^2 - a^2).$$

Hence

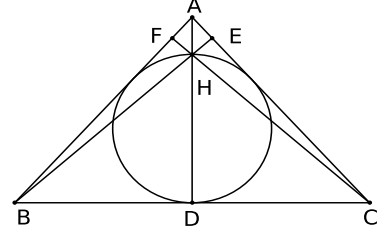
$$r^2 = \frac{[ABC]^2}{s^2} = \frac{a^2(b^2 - a^2)}{(b + a)^2}.$$

Using this we get

$$\frac{a^4}{b^2 - a^2} = 4 \left( \frac{a^2(b^2 - a^2)}{(b + a)^2} \right).$$

Therefore  $a^2 = 4(b - a)^2$ , which gives  $a = 2(b - a)$  or  $2b = 3a$ . Finally,

$$\frac{AB}{BC} = \frac{b}{2a} = \frac{3}{4}.$$



**Alternate Solution 1:**

We use the known facts  $BH = 2R \cos B$  and  $r = 4R \sin(A/2) \sin(B/2) \sin(C/2)$ , where  $R$  is the circumradius of  $\triangle ABC$  and  $r$  its in-radius. Therefore

$$HD = BH \sin \angle HBD = 2R \cos B \sin \left( \frac{\pi}{2} - C \right) = 2R \cos^2 B,$$

since  $\angle C = \angle B$ . But  $\angle B = (\pi - \angle A)/2$ , since  $ABC$  is isosceles. Thus we obtain

$$HD = 2R \cos^2 \left( \frac{\pi}{2} - \frac{A}{2} \right).$$

However  $HD$  is also the diameter of the in circle. Therefore  $HD = 2r$ . Thus we get

$$2R \cos^2 \left( \frac{\pi}{2} - \frac{A}{2} \right) = 2r = 8R \sin(A/2) \sin^2((\pi - A)/4).$$

This reduces to

$$\sin(A/2) = 2(1 - \sin(A/2)).$$

Therefore  $\sin(A/2) = 2/3$ . We also observe that  $\sin(A/2) = BD/AB$ . Finally

$$\frac{AB}{BC} = \frac{AB}{2BD} = \frac{1}{2 \sin(A/2)} = \frac{3}{4}.$$

**Alternate Solution 2:**

Let  $D$  be the mid-point of  $BC$ . Extend  $AD$  to meet the circumcircle in  $L$ . Then we know that  $HD = DL$ . But  $HD = 2r$ . Thus  $DL = 2r$ . Therefore  $IL = ID + DL = r + 2r = 3r$ . We also know that  $LB = LI$ . Therefore  $LB = 3r$ . This gives

$$\frac{BL}{LD} = \frac{3r}{2r} = \frac{3}{2}.$$

But  $\triangle BLD$  is similar to  $\triangle ABD$ . So

$$\frac{AB}{BD} = \frac{BL}{LD} = \frac{3}{2}.$$

Finally,

$$\frac{AB}{BC} = \frac{AB}{2BD} = \frac{3}{4}.$$

**Alternate Solution 3:**

Let  $D$  be the mid-point of  $BC$  and  $E$  be the mid-point of  $DC$ . Since  $DI = IH (= r)$  and  $DE = EC$ , the mid-point theorem implies that  $IE \parallel CH$ . But  $CH \perp AB$ . Therefore  $EI \perp AB$ . Let  $EI$  meet  $AB$  in  $F$ . Then  $F$  is the point of tangency of the incircle of  $\triangle ABC$  with  $AB$ . Since the incircle is also tangent to  $BC$  at  $D$ , we have  $BF = BD$ . Observe that  $\triangle BFE$  is similar to  $\triangle BDA$ . Hence

$$\frac{AB}{BD} = \frac{BE}{BF} = \frac{BE}{BD} = \frac{BD + DE}{BD} = 1 + \frac{DE}{BD} = \frac{3}{2}.$$

This gives

$$\frac{AB}{BC} = \frac{3}{4}.$$

2. For positive real numbers  $a, b, c$ , which of the following statements necessarily implies  $a = b = c$ : (I)  $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$ , (II)  $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$ ? Justify your answer.

**Solution:** We show that (I) need not imply that  $a = b = c$  where as (II) always implies  $a = b = c$ .

Observe that  $a(b^3 + c^3) = b(c^3 + a^3)$  gives  $c^3(a - b) = ab(a^2 - b^2)$ . This gives either  $a = b$  or  $ab(a + b) = c^3$ . Similarly,  $b = c$  or  $bc(b + c) = a^3$ . If  $a \neq b$  and  $b \neq c$ , we obtain

$$ab(a + b) = c^3, \quad bc(b + c) = a^3.$$

Therefore

$$b(a^2 - c^2) + b^2(a - c) = c^3 - a^3.$$

This gives  $(a - c)(a^2 + b^2 + c^2 + ab + bc + ca) = 0$ . Since  $a, b, c$  are positive, the only possibility is  $a = c$ . We have therefore 4 possibilities:  $a = b = c$ ;  $a \neq b$ ,  $b \neq c$  and  $c = a$ ;  $b \neq c$ ,  $c \neq a$  and  $a = b$ ;  $c \neq a$ ,  $a \neq b$  and  $b = c$ .

Suppose  $a = b$  and  $b, a \neq c$ . Then  $b(c^3 + a^3) = c(a^3 + b^3)$  gives  $ac^3 + a^4 = 2ca^3$ . This implies that  $a(a - c)(a^2 - ac - c^2) = 0$ . Therefore  $a^2 - ac - c^2 = 0$ . Putting  $a/c = x$ , we get the quadratic equation  $x^2 - x - 1 = 0$ . Hence  $x = (1 + \sqrt{5})/2$ . Thus we get

$$a = b = \left( \frac{1 + \sqrt{5}}{2} \right) c, \quad c \text{ arbitrary positive real number.}$$

Similarly, we get other two cases:

$$b = c = \left( \frac{1 + \sqrt{5}}{2} \right) a, \quad a \text{ arbitrary positive real number;}$$

$$c = a = \left( \frac{1 + \sqrt{5}}{2} \right) b, \quad b \text{ arbitrary positive real number.}$$

And  $a = b = c$  is the fourth possibility.

Consider (II):  $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$ . Suppose  $a, b, c$  are mutually distinct. We may assume  $a = \max\{a, b, c\}$ . Hence  $a > b$  and  $a > c$ . Using  $a > b$ , we get from the first relation that  $a^3 + b^3 < b^3 + c^3$ . Therefore  $a^3 < c^3$  forcing  $a < c$ . This contradicts  $a > c$ . We conclude that  $a, b, c$  cannot be mutually distinct. This means some two must be equal. If  $a = b$ , the equality of the first two expressions give  $a^3 + b^3 = b^3 + c^3$  so that  $a = c$ . Similarly, we can show that  $b = c$  implies  $b = a$  and  $c = a$  gives  $c = b$ .

**Alternate for (II) by a contestant:** We can write

$$\begin{aligned} \frac{a^3}{c} + \frac{b^3}{c} &= \frac{c^3}{a} + a^2, \\ \frac{b^3}{a} + \frac{c^3}{a} &= \frac{a^3}{b} + b^2, \\ \frac{c^3}{b} + \frac{a^3}{b} &= \frac{b^3}{c} + c^2. \end{aligned}$$

Adding, we get

$$\frac{a^3}{c} + \frac{b^3}{a} + \frac{c^3}{b} = a^2 + b^2 + c^2.$$

Using C-S inequality, we have

$$\begin{aligned}
 (a^2 + b^2 + c^2)^2 &= \left( \frac{\sqrt{a^3}}{\sqrt{c}} \cdot \sqrt{ac} + \frac{\sqrt{b^3}}{\sqrt{a}} \cdot \sqrt{ba} + \frac{\sqrt{c^3}}{\sqrt{b}} \cdot \sqrt{cb} \right)^2 \\
 &\leq \left( \frac{a^3}{c} + \frac{b^3}{a} + \frac{c^3}{b} \right) (ac + ba + cb) \\
 &= (a^2 + b^2 + c^2)(ab + bc + ca).
 \end{aligned}$$

Thus we obtain

$$a^2 + b^2 + c^2 \leq ab + bc + ca.$$

However this implies  $(a - b)^2 + (b - c)^2 + (c - a)^2 \leq 0$  and hence  $a = b = c$ .

3. Let  $\mathbb{N}$  denote the set of all natural numbers. Define a function  $T : \mathbb{N} \rightarrow \mathbb{N}$  by  $T(2k) = k$  and  $T(2k + 1) = 2k + 2$ . We write  $T^2(n) = T(T(n))$  and in general  $T^k(n) = T^{k-1}(T(n))$  for any  $k > 1$ .
- (i) Show that for each  $n \in \mathbb{N}$ , there exists  $k$  such that  $T^k(n) = 1$ .
- (ii) For  $k \in \mathbb{N}$ , let  $c_k$  denote the number of elements in the set  $\{n : T^k(n) = 1\}$ . Prove that  $c_{k+2} = c_{k+1} + c_k$ , for  $k \geq 1$ .

**Solution:**

(i) For  $n = 1$ , we have  $T(1) = 2$  and  $T^2(1) = T(2) = 1$ . Hence we may assume that  $n > 1$ .

Suppose  $n > 1$  is even. Then  $T(n) = n/2$ . We observe that  $(n/2) \leq n - 1$  for  $n > 1$ .

Suppose  $n > 1$  is odd so that  $n \geq 3$ . Then  $T(n) = n + 1$  and  $T^2(n) = (n + 1)/2$ . Again we see that  $(n + 1)/2 \leq (n - 1)$  for  $n \geq 3$ .

Thus we see that in at most  $2(n - 1)$  steps  $T$  sends  $n$  to 1. Hence  $k \leq 2(n - 1)$ . (Here  $2(n - 1)$  is only a bound. In reality, less number of steps will do.)

(ii) We show that  $c_n = f_{n+1}$ , where  $f_n$  is the  $n$ -th Fibonacci number.

Let  $n \in \mathbb{N}$  and let  $k \in \mathbb{N}$  be such that  $T^k(n) = 1$ . Here  $n$  can be odd or even. If  $n$  is even, it can be either of the form  $4d + 2$  or of the form  $4d$ .

If  $n$  is odd, then  $1 = T^k(n) = T^{k-1}(n + 1)$ . (Observe that  $k > 1$ ; otherwise we get  $n + 1 = 1$  which is impossible since  $n \in \mathbb{N}$ .) Here  $n + 1$  is even.

If  $n = 4d + 2$ , then again  $1 = T^k(4d + 2) = T^{k-1}(2d + 1)$ . Here  $2d + 1 = n/2$  is odd.

Thus each solution of  $T^{k-1}(m) = 1$  produces exactly one solution of  $T^k(n) = 1$  and  $n$  is either odd or of the form  $4d + 2$ .

If  $n = 4d$ , we see that  $1 = T^k(4d) = T^{k-1}(2d) = T^{k-2}(d)$ . This shows that each solution of  $T^{k-2}(m) = 1$  produces exactly one solution of  $T^k(n) = 1$  of the form  $4d$ .

Thus the number of solutions of  $T^k(n) = 1$  is equal to the number of solutions of  $T^{k-1}(m) = 1$  and the number of solutions of  $T^{k-2}(l) = 1$  for  $k > 2$ . This shows that  $c_k = c_{k-1} + c_{k-2}$  for  $k > 2$ . We also observe that 2 is the only number which goes to 1 in one step and 4 is the only number which goes to 1 in two steps. Hence  $c_1 = 1$  and  $c_2 = 2$ . This proves that  $c_n = f_{n+1}$  for all  $n \in \mathbb{N}$ .

4. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number  $n \geq 3$ , prove that there is a regular  $n$ -sided polygon all of whose vertices are blue.

**Solution:** Let  $A_1, A_2, \dots, A_{2016}$  be 2016 points on the circle which are coloured *red* and the remain-

ing blue. Let  $n \geq 3$  and let  $B_1, B_2, \dots, B_n$  be a regular  $n$ -sided polygon inscribed in this circle with the vertices chosen in anti-clock-wise direction. We place  $B_1$  at  $A_1$ . (It is possible, in this position, some other  $B$ 's also coincide with some other  $A$ 's.) Rotate the polygon in anti-clock-wise direction gradually till some  $B$ 's coincide with (an equal number of)  $A$ 's second time. We again rotate the polygon in the same direction till some  $B$ 's coincide with an equal number of  $A$ 's third time, and so on until we return to the original position, i.e.,  $B_1$  at  $A_1$ . We see that the number of rotations will not be more than  $2016 \times n$ , that is, at most these many times some  $B$ 's would have coincided with an equal number of  $A$ 's. Since the interval  $(0, 360^\circ)$  has infinitely many points, we can find a value  $\alpha^\circ \in (0, 360^\circ)$  through which the polygon can be rotated from its initial position such that no  $B$  coincides with any  $A$ . This gives a  $n$ -sided regular polygon having only blue vertices.

**Alternate Solution:** Consider a regular  $2017 \times n$ -gon on the circle; say,  $A_1 A_2 A_3 \dots A_{2017n}$ . For each  $j$ ,  $1 \leq j \leq 2017$ , consider the points  $\{A_k : k \equiv j \pmod{2017}\}$ . These are the vertices of a regular  $n$ -gon, say  $S_j$ . We get 2017 regular  $n$ -gons;  $S_1, S_2, \dots, S_{2017}$ . Since there are only 2016 red points, by pigeon-hole principle there must be some  $n$ -gon among these 2017 which does not contain any red point. But then it is a blue  $n$ -gon.

5. Let  $ABC$  be a right-angled triangle with  $\angle B = 90^\circ$ . Let  $D$  be a point on  $AC$  such that the in-radii of the triangles  $ABD$  and  $CBD$  are equal. If this common value is  $r'$  and if  $r$  is the in-radius of triangle  $ABC$ , prove that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}.$$

**Solution:** Let  $E$  and  $F$  be the incentres of triangles  $ABD$  and  $CBD$  respectively. Let the incircles of triangles  $ABD$  and  $CBD$  touch  $AC$  in  $P$  and  $Q$  respectively. If  $\angle BDA = \theta$ , we see that

$$r' = PD \tan(\theta/2) = QD \cot(\theta/2).$$

Hence

$$PQ = PD + QD = r' \left( \cot \frac{\theta}{2} + \tan \frac{\theta}{2} \right) = \frac{2r'}{\sin \theta}.$$

But we observe that

$$DP = \frac{BD + DA - AB}{2}, \quad DQ = \frac{BD + DC - BC}{2}.$$

Thus  $PQ = (b - c - a + 2BD)/2$ . We also have

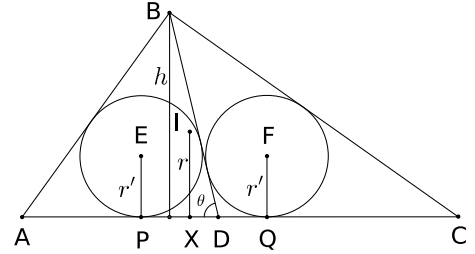
$$\begin{aligned} \frac{ac}{2} &= [ABC] = [ABD] + [CBD] = r' \frac{(AB + BD + DA)}{2} + r' \frac{(CB + BD + DC)}{2} \\ &= r' \frac{(c + a + b + 2BD)}{2} = r'(s + BD). \end{aligned}$$

But

$$r' = \frac{PQ \sin \theta}{2} = \frac{PQ \cdot h}{2BD},$$

where  $h$  is the altitude from  $B$  on to  $AC$ . But we know that  $h = ac/b$ . Thus we get

$$ac = 2 \times r'(s + BD) = 2 \times \frac{PQ \cdot h}{2 \times BD} (s + BD) = \frac{(b - c - a + 2BD)ca(s + BD)}{2 \times BD \times b}.$$



Thus we get

$$2 \times BD \times b = 2 \times (BD - (s - b))(s + BD).$$

This gives  $BD^2 = s(s - b)$ . Since  $ABC$  is a right-angled triangle  $r = s - b$ . Thus we get  $BD^2 = rs$ . On the other hand, we also have  $[ABC] = r'(s + BD)$ . Thus we get

$$rs = [ABC] = r'(s + BD).$$

Hence

$$\frac{1}{r'} = \frac{1}{r} + \frac{BD}{rs} = \frac{1}{r} + \frac{1}{BD}.$$

**Alternate Solution 1:** Observe that

$$\frac{r'}{r} = \frac{AP}{AX} = \frac{CQ}{CX} = \frac{AP + CQ}{AC},$$

where  $X$  is the point at which the incircle of  $ABC$  touches the side  $AC$ . If  $s_1$  and  $s_2$  are respectively the semi-perimeters of triangles  $ABD$  and  $CBD$ , we know  $AP = s_1 - BD$  and  $CQ = s_2 - BD$ . Therefore

$$\frac{r'}{r} = \frac{(s_1 - BD) + (s_2 - BD)}{AC} = \frac{s_1 + s_2 - 2BD}{b}.$$

But

$$s_1 + s_2 = \frac{AD + BD + c}{2} + \frac{CD + BD + a}{2} = \frac{(a + b + c) + 2BD}{2} = \frac{s + BD}{2}.$$

This gives

$$\frac{r'}{r} = \frac{s + BD - 2BD}{b} = \frac{s - BD}{b}.$$

We also have

$$r' = \frac{[ABD]}{s_1} = \frac{[CBD]}{s_2} = \frac{[ABD] + [CBD]}{s_1 + s_2} = \frac{[ABC]}{s + BD} = \frac{rs}{s + BD}.$$

This implies that

$$\frac{r'}{r} = \frac{s}{s + BD}.$$

Comparing the two expressions for  $r'/r$ , we see that

$$\frac{s - BD}{b} = \frac{s}{s + BD}.$$

Therefore  $s^2 - BD^2 = bs$ , or  $BD^2 = s(s - b)$ . Thus we get  $BD = \sqrt{s(s - b)}$ .

We know now that

$$\frac{r'}{r} = \frac{s}{s + BD} = \frac{s - BD}{b} = \frac{BD}{(s - b) + BD} = \frac{\sqrt{s(s - b)}}{(s - b) + \sqrt{s(s - b)}} = \frac{\sqrt{s}}{\sqrt{s - b} + \sqrt{s}}.$$

Therefore

$$\frac{r}{r'} = 1 + \sqrt{\frac{s - b}{s}}.$$

This gives

$$\frac{1}{r'} = \frac{1}{r} + \left( \sqrt{\frac{s - b}{s}} \right) \frac{1}{r}.$$

But

$$\left(\sqrt{\frac{s-b}{s}}\right) \frac{1}{r} = \left(\frac{s-b}{\sqrt{s(s-b)}}\right) \frac{1}{r} = \left(\frac{s-b}{BD}\right) \frac{1}{r}.$$

If  $\angle B = 90^\circ$ , we know that  $r = s - b$ . Therefore we get

$$\frac{1}{r'} = \frac{1}{r} + \left(\frac{s-b}{BD}\right) \frac{1}{r} = \frac{1}{r} + \frac{1}{BD}.$$

**Alternate Solution 2 by a contestant:** Observe that  $\angle EDF = 90^\circ$ . Hence  $\triangle EDP$  is similar to  $\triangle DFQ$ . Therefore  $DP \cdot DQ = EP \cdot FQ$ . Taking  $DP = y_1$  and  $DQ = x_1$ , we get  $x_1 y_1 = (r')^2$ . We also observe that  $BD = x_1 + x_2 = y_1 + y_2$ . Since  $\angle EBF = 45^\circ$ , we get

$$1 = \tan 45^\circ = \tan(\beta_1 + \beta_2) = \frac{\tan \beta_1 + \tan \beta_2}{1 - \tan \beta_1 \tan \beta_2}.$$

But  $\tan \beta_1 = r'/y_2$  and  $\tan \beta_2 = r'/x_2$ . Hence we obtain

$$1 = \frac{(r'/y_2) + (r'/x_2)}{1 - (r')^2/x_2 y_2}.$$

Solving for  $r'$ , we get

$$r' = \frac{x_2 y_2 - x_1 y_1}{x_2 + y_2}.$$

We also know

$$r = \frac{AB + BC - AC}{2} = \frac{x_2 + y_2 - (x_1 + y_1)}{2} = \frac{(x_2 - x_1) + (y_2 - y_1)}{2}.$$

Finally,

$$\begin{aligned} \frac{1}{r} + \frac{1}{BD} &= \frac{2}{(x_2 - x_1) + (y_2 - y_1)} + \frac{1}{x_1 + x_2} \\ &= \frac{2x_1 + 2x_2 + (x_2 - x_1) + (y_2 - y_1)}{(x_1 + x_2)((x_2 - x_1) + (y_2 - y_1))}. \end{aligned}$$

But we can write

$$2x_1 + 2x_2 + (x_2 - x_1) + (y_2 - y_1) = (x_1 + x_2 + x_2 - x_1) + (y_1 + y_2 + y_2 - y_1) = 2(x_2 + y_2),$$

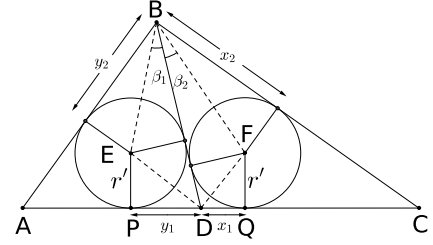
and

$$\begin{aligned} (x_1 + x_2)((x_2 - x_1) + (y_2 - y_1)) &= 2(x_1 + x_2)(x_2 - y_1) \\ &= 2(x_2(x_2 + x_1 - y_1) - x_1 y_1) = 2(x_2 y_2 - x_1 y_1). \end{aligned}$$

Therefore

$$\frac{1}{r} + \frac{1}{BD} = \frac{2(x_2 + y_2)}{2(x_2 y_2 - x_1 y_1)} = \frac{1}{r'}.$$

**Remark:** One can also choose  $B = (0, 0)$ ,  $A = (0, a)$  and  $C = (1, 0)$  and the coordinate geometry proof gets reduced considerably.



6. Consider a non-constant arithmetic progression  $a_1, a_2, \dots, a_n, \dots$ . Suppose there exist relatively prime positive integers  $p > 1$  and  $q > 1$  such that  $a_1^2$ ,  $a_{p+1}^2$  and  $a_{q+1}^2$  are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

**Solution:** Let us take  $a_1 = a$ . We have

$$a^2 = a + kd, \quad (a + pd)^2 = a + ld, \quad (a + qd)^2 = a + md.$$

Thus we have

$$a + ld = (a + pd)^2 = a^2 + 2pad + p^2d^2 = a + kd + 2pad + p^2d^2.$$

Since we have non-constant AP, we see that  $d \neq 0$ . Hence we obtain  $2pa + p^2d = l - k$ . Similarly, we get  $2qa + q^2d = m - k$ . Observe that  $p^2q - pq^2 \neq 0$ . Otherwise  $p = q$  and  $\gcd(p, q) = p > 1$  which is a contradiction to the given hypothesis that  $\gcd(p, q) = 1$ . Hence we can solve the two equations for  $a, d$ :

$$a = \frac{p^2(m - k) - q^2(l - k)}{2(p^2q - pq^2)}, \quad d = \frac{q(l - k) - p(m - k)}{p^2q - pq^2}.$$

It follows that  $a, d$  are rational numbers. We also have

$$p^2a^2 = p^2a + kp^2d.$$

But  $p^2d = l - k - 2pa$ . Thus we get

$$p^2a^2 = p^2a + k(l - k - 2pa) = (p - 2k)pa + k(l - k).$$

This shows that  $pa$  satisfies the equation

$$x^2 - (p - 2k)x - k(l - k) = 0.$$

Since  $a$  is rational, we see that  $pa$  is rational. Write  $pa = w/z$ , where  $w$  is an integer and  $z$  is a natural numbers such that  $\gcd(w, z) = 1$ . Substituting in the equation, we obtain

$$w^2 - (p - 2k)wz - k(l - k)z^2 = 0.$$

This shows  $z$  divides  $w$ . Since  $\gcd(w, z) = 1$ , it follows that  $z = 1$  and  $pa = w$  an integer. (In fact any rational solution of a monic polynomial with integer coefficients is necessarily an integer.) Similarly, we can prove that  $qa$  is an integer. Since  $\gcd(p, q) = 1$ , there are integers  $u$  and  $v$  such that  $pu + qv = 1$ . Therefore  $a = (pa)u + (qa)v$ . It follows that  $a$  is an integer.

But  $p^2d = l - k - 2pa$ . Hence  $p^2d$  is an integer. Similarly,  $q^2d$  is also an integer. Since  $\gcd(p^2, q^2) = 1$ , it follows that  $d$  is an integer. Combining these two, we see that all the terms of the AP are integers.

**Alternatively**, we can prove that  $a$  and  $d$  are integers in another way. We have seen that  $a$  and  $d$  are rationals; and we have three relations:

$$a^2 = a + kd, \quad p^2d + 2pa = n_1, \quad q^2d + 2qa = n_2,$$

where  $n_1 = l - k$  and  $n_2 = m - k$ . Let  $a = u/v$  and  $d = x/y$  where  $u, x$  are integers and  $v, y$  are natural numbers, and  $\gcd(u, v) = 1$ ,  $\gcd(x, y) = 1$ . Putting this in these relations, we obtain

$$u^2y = uvx + kxv^2, \tag{1}$$

$$2puy + p^2vx = vyn_1, \tag{2}$$

$$2quy + q^2vx = vyn_2. \tag{3}$$

Now (1) shows that  $v|u^2y$ . Since  $\gcd(u, v) = 1$ , it follows that  $v|y$ . Similarly (2) shows that  $y|p^2vx$ . Using  $\gcd(y, x) = 1$ , we get that  $y|p^2v$ . Similarly, (3) shows that  $y|q^2v$ . Therefore  $y$  divides  $\gcd(p^2v, q^2v) = v$ . The two results  $v|y$  and  $y|v$  imply  $v = y$ , since both  $v, y$  are positive.

Substitute this in (1) to get

$$u^2 = uv + kxv.$$

This shows that  $v|u^2$ . Since  $\gcd(u, v) = 1$ , it follows that  $v = 1$ . This gives  $v = y = 1$ . Finally  $a = u$  and  $d = x$  which are integers.

————000000————

# 32<sup>nd</sup> Indian National Mathematical Olympiad-2017

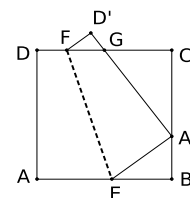
Time: 4 hours

January 15, 2017

## Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks: 102.
- Answer all the questions.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. In the given figure,  $ABCD$  is a square sheet of paper. It is folded along  $EF$  such that  $A$  goes to a point  $A'$  different from  $B$  and  $C$ , on the side  $BC$  and  $D$  goes to  $D'$ . The line  $A'D'$  cuts  $CD$  in  $G$ . Show that the inradius of the triangle  $GCA'$  is the sum of the inradii of the triangles  $GD'F$  and  $A'BE$ .



2. Suppose  $n \geq 0$  is an integer and all the roots of  $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$  are integers. Find all possible values of  $\alpha$ .
3. Find the number of triples  $(x, a, b)$  where  $x$  is a real number and  $a, b$  belong to the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  such that

$$x^2 - a\{x\} + b = 0,$$

where  $\{x\}$  denotes the fractional part of the real number  $x$ . (For example  $\{1.1\} = 0.1 = \{-0.9\}$ .)

4. Let  $ABCDE$  be a convex pentagon in which  $\angle A = \angle B = \angle C = \angle D = 120^\circ$  and side lengths are five *consecutive integers* in some order. Find all possible values of  $AB + BC + CD$ .
5. Let  $ABC$  be a triangle with  $\angle A = 90^\circ$  and  $AB < AC$ . Let  $AD$  be the altitude from  $A$  on to  $BC$ . Let  $P, Q$  and  $I$  denote respectively the incentres of triangles  $ABD$ ,  $ACD$  and  $ABC$ . Prove that  $AI$  is perpendicular to  $PQ$  and  $AI = PQ$ .
6. Let  $n \geq 1$  be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \dots$$

Show that  $2x - 1, 2x, 2x + 1$  form the sides of a triangle whose area and inradius are also integers.

—————000000—————

# 32वाँ भारतीय राष्ट्रीय गणित ओलिंपियाड - 2017

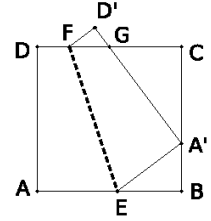
समय: 4 घंटा

जनवरी 15, 2017

निर्देश :

- किसी भी तरह के गणक (calculators) तथा चांदा (protractors) के प्रयोग की अनुमति नहीं है.
- पैमाना (rulers) तथा परकार (compasses) के प्रयोग की अनुमति है.
- सभी प्रश्नों के अंक एकसमान हैं. अधिकतम अंक : 102.
- सभी प्रश्नों के उत्तर दीजिये.
- प्रत्येक प्रश्न का उत्तर नए पेज से प्रारंभ कीजिये. प्रश्न क्रमांक स्पष्ट रूप से इंगित कीजिये.

1. दिए हुए चित्र में  $ABCD$  एक वर्गाकार कागज है. इसे  $EF$  के परितः इस प्रकार मोड़ा जाता है कि  $A$ , बिंदु  $B$  तथा  $C$  से भिन्न भुजा  $BC$  पर बिंदु  $A'$  पर आ जाता है तथा  $D$ , बिंदु  $D'$  पर जाता है. रेखा  $A'D'$ ,  $CD$  को  $G$  पर काटती है. दिखाइये कि त्रिभुज  $GCA'$  की अंतःत्रिज्या त्रिभुजों  $GD'F$  तथा  $A'BE$  की अंतःत्रिज्याओं के योग के बराबर है.



2. मान लीजिये कि  $n \geq 0$  एक पूर्णांक है तथा  $x^3 + ax + 4 - (2 \times 2016^n) = 0$  के सभी मूल पूर्णांक हैं.  $\alpha$  के सभी संभावित मान ज्ञात कीजिये.
3. उन सभी त्रियुग्मों  $(x, a, b)$  की संख्याएं ज्ञात कीजिये जिसमें कि  $x$  एक वास्तविक संख्या है तथा  $a, b$  समुच्चय  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  से इस प्रकार संबंधित है कि
- $$x^2 - a\{x\} + b = 0$$
- जहाँ  $\{x\}$  वास्तविक संख्या  $x$  का भिन्नात्मक भाग है. (उदाहरण के लिए  $\{1.1\} = 0.1 = \{-0.9\}$ .)
4. मान लीजिये कि  $ABCDE$  एक उत्तल पंचभुज (convex pentagon) है जिसमें  $\angle A = \angle B = \angle C = \angle D = 120^\circ$  है तथा भुजाओं की लम्बाई किसी क्रम में पांच क्रमागत पूर्णांक (consecutive integer) हैं.  $AB + BC + CD$  के सभी संभव मान ज्ञात कीजिये.
5. मान लीजिये कि  $ABC$  एक त्रिभुज है जिसमें  $\angle A = 90^\circ$  तथा  $AB < AC$ . मान लीजिये कि  $AD$ ,  $A$  से  $BC$  पर शीर्षलम्ब है. मान लीजिये कि  $P, Q$  तथा  $I$  क्रमशः त्रिभुज  $ABD, ACD$  तथा  $ABC$  के अंतःकेंद्रों को निरूपित करते हैं. सिद्ध कीजिये कि  $AI$ ,  $PQ$  के लम्बवत है तथा  $AI = PQ$ .
6. मान लीजिये कि  $n \geq 1$  एक पूर्णांक है. निम्न योग पर विचार कीजिये

$$x = \sum_{k=0}^n \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \dots$$

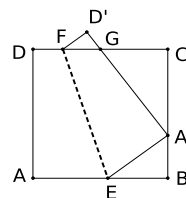
दिखाइए कि  $2x - 1$ ,  $2x$ ,  $2x + 1$  उस त्रिभुज की भुजाओं को बनाते हैं जिसका क्षेत्रफल तथा अंतःत्रिज्या भी एक पूर्णांक हैं.

-----000000-----

# 32<sup>nd</sup> Indian National Mathematical Olympiad-2017

## Problems and Solutions

1. In the given figure,  $ABCD$  is a square paper. It is folded along  $EF$  such that  $A$  goes to a point  $A' \neq C, B$  on the side  $BC$  and  $D$  goes to  $D'$ . The line  $A'D'$  cuts  $CD$  in  $G$ . Show that the inradius of the triangle  $GCA'$  is the sum of the inradii of the triangles  $GD'F$  and  $A'BE$ .



**Solution:** Observe that the triangles  $GCA'$  and  $A'BE$  are similar to the triangle  $GD'F$ . If  $GF = u$ ,  $GD' = v$  and  $D'F = w$ , then we have

$$A'G = pu, CG = pv, A'C = pw, \quad A'E = qu, BE = qw, A'B = qv.$$

If  $r$  is the inradius of  $\triangle GD'F$ , then  $pr$  and  $qr$  are respectively the inradii of triangles  $GCA'$  and  $A'BE$ . We have to show that  $pr = r + qr$ . We also observe that

$$AE = EA', \quad DF = FD'.$$

Therefore

$$pw + qv = qw + qu = w + u + pv = v + pu.$$

The last two equalities give  $(p-1)(u-v) = w$ . The first two equalities give  $(p-q)w = q(u-v)$ . Hence

$$\frac{p-q}{q} = \frac{u-v}{w} = \frac{1}{p-1}.$$

This simplifies to  $p(p-q-1) = 0$ . Since  $p \neq 0$ , we get  $p = q+1$ . This implies that  $pr = qr + r$ .

2. Suppose  $n \geq 0$  is an integer and all the roots of  $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$  are integers. Find all possible values of  $\alpha$ .

**Solution 1:** Let  $u, v, w$  be the roots of  $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ . Then  $u + v + w = 0$  and  $uvw = -4 + (2 \times 2016^n)$ . Therefore we obtain

$$uv(u+v) = 4 - (2 \times 2016^n).$$

Suppose  $n \geq 1$ . Then we see that  $uv(u+v) \equiv 4 \pmod{2016^n}$ . Therefore  $uv(u+v) \equiv 1 \pmod{3}$  and  $uv(u+v) \equiv 1 \pmod{9}$ . This implies that  $u \equiv 2 \pmod{3}$  and  $v \equiv 2 \pmod{3}$ . This shows that modulo 9 the pair  $(u, v)$  could be any one of the following:

$$(2, 2), (2, 5), (2, 8), (5, 2), (5, 5), (5, 8), (8, 2), (8, 5), (8, 8).$$

In each case it is easy to check that  $uv(u+v) \not\equiv 4 \pmod{9}$ . Hence  $n = 0$  and  $uv(u+v) = 2$ . It follows that  $(u, v) = (1, 1), (1, -2)$  or  $(-2, 1)$ . Thus

$$\alpha = uv + vw + wu = uv - (u+v)^2 = -3$$

for every pair  $(u, v)$ .

**Solution 2:** Let  $a, b, c \in \mathbb{Z}$  be the roots of the given equation for some  $n \in \mathbb{N}_0$ . By Vieta Theorem, we know that

$$a + b + c = 0$$

$$ab + bc + ca = \alpha$$

$$abc = 2 \times 2016^n - 4$$

If possible, let us have  $n \geq 1$ . Since  $7|2016$ , we have that

$$7|abc + 4 \implies 7|3(abc + 4) \implies 7|3abc + 12 \implies 7|3abc + 5$$

Since we have  $a + b + c = 0$ , we get that  $3abc = a^3 + b^3 + c^3$ . Substituting this in the earlier expression, we get that

$$a^3 + b^3 + c^3 + 5 \equiv 0 \pmod{7}$$

Consider below, a table calculating the residues of cubes modulo 7.

$x$	0	1	2	3	4	5	6
$x^3$	0	1	1	-1	1	-1	-1

Hence, we know that if  $x \in \mathbb{N}$ , then we have  $x^3 \equiv 0, 1, -1 \pmod{7}$ . Since  $a^3 + b^3 + c^3 \equiv 2 \pmod{7}$ , we see that we must have one of the numbers divisible by 7 and the other two numbers, when cubed, must leave 1 as remainder modulo 7. Without of generality, let us assume that

$$a \equiv 0 \pmod{7}, \quad b^3, c^3 \equiv 1 \pmod{7}$$

Hence, we have  $b, c \equiv 1, 2, 4 \pmod{7}$ . We will consider all possible values of  $b + c$  modulo 7. Since the expression is symmetric in  $b, c$ , modulo 7, we will consider  $b \leq c$ .

$b$	1	1	1	2	2	4
$c$	1	2	4	2	4	4
$b + c$	2	3	5	4	6	1

We see that, in all the above cases, we get  $7 \nmid b + c$ . But this is a contradiction, since  $7|a + b + c$  and  $7|a$  together imply that  $7|b + c$ . Hence, we cannot have  $n \geq 1$ . Hence, the only possible value is  $n = 0$ . Substituting this value in the original equation, the equation becomes

$$x^3 + \alpha x + 2 = 0$$

Solving the equations  $a + b + c = 0$  and  $abc = -2$  in integers, we see that the only possible solutions  $(a, b, c)$  are permutations of  $(1, 1, -2)$ . In case of any permutation,  $\alpha = -3$ . Substituting this value of  $\alpha$  back in the equation, we see that we indeed, get integer roots. Hence, the only possible value for  $\alpha$  is  $-3$ .

3. Find the number of triples  $(x, a, b)$  where  $x$  is a real number and  $a, b$  belong to the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  such that

$$x^2 - a\{x\} + b = 0,$$

where  $\{x\}$  denotes the fractional part of the real number  $x$ . (For example  $\{1.1\} = 0.1 = \{-0.9\}$ .)

**Solution:** Let us write  $x = n + f$  where  $n = [x]$  and  $f = \{x\}$ . Then

$$f^2 + (2n - a)f + n^2 + b = 0. \quad (1)$$

Observe that the product of the roots of (1) is  $n^2 + b \geq 1$ . If this equation has to have a solution  $0 \leq f < 1$ , the larger root of (1) is greater 1. We conclude that the equation (1) has a real root less than 1 only if  $P(1) < 0$  where  $P(y) = y^2 + (2n - a)y + n^2 + b$ . This gives

$$1 + 2n - a + n^2 + 2b < 0.$$

Therefore we have  $(n + 1)^2 + b < a$ . If  $n \geq 2$ , then  $(n + 1)^2 + b \geq 10 > a$ . Hence  $n \leq 1$ . If  $n \leq -4$ , then again  $(n + 1)^2 + b \geq 10 > a$ . Thus we have the range for  $n$ :  $-3, -2, -1, 0, 1$ .

If  $n = -3$  or  $n = 1$ , we have  $(n + 1)^2 = 4$ . Thus we must have  $4 + b < a$ . If  $a = 9$ , we must have  $b = 4, 3, 2, 1$  giving 4 values. For  $a = 8$ , we must have  $b = 3, 2, 1$  giving 3 values. Similarly, for  $a = 7$  we get 2 values of  $b$  and  $a = 6$  leads to 1 value of  $b$ . In each case we get a real value of  $f < 1$  and this leads to a solution for  $x$ . Thus we get totally  $2(4 + 3 + 2 + 1) = 20$  values of the triple  $(x, a, b)$ .

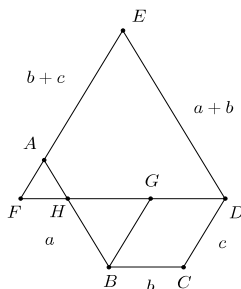
For  $n = -2$  and  $n = 0$ , we have  $(n + 1)^2 = 1$ . Hence we require  $1 + b < a$ . We again count pairs  $(a, b)$  such that  $a - b > 1$ . For  $a = 9$ , we get 7 values of  $b$ ; for  $a = 8$  we get 6 values of  $b$  and so on. Thus we get  $2(7 + 6 + 5 + 4 + 3 + 2 + 1) = 56$  values for the triple  $(x, a, b)$ .

Suppose  $n = -1$  so that  $(n + 1)^2 = 0$ . In this case we require  $b < a$ . We get  $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$  values for the triple  $(x, a, b)$ .

Thus the total number of triples  $(x, a, b)$  is  $20 + 56 + 36 = 112$ .

4. Let  $ABCDE$  be a convex pentagon in which  $\angle A = \angle B = \angle C = \angle D = 120^\circ$  and whose side lengths are 5 consecutive integers in some order. Find all possible values of  $AB + BC + CD$ .

**Solution 1:** Let  $AB = a$ ,  $BC = b$ , and  $CD = c$ . By symmetry, we may assume that  $c < a$ . We show that  $DE = a + b$  and  $EA = b + c$ .



Draw a line parallel to  $BC$  through  $D$ . Extend  $EA$  to meet this line at  $F$ . Draw a line parallel to  $CD$  through  $B$  and let it intersect  $DF$  in  $G$ . Let  $AB$  intersect  $DF$  in  $H$ . We have  $\angle FDE = 60^\circ$  and  $\angle E = 60^\circ$ . Hence  $EFD$  is an equilateral triangle. Similarly  $AFH$  and  $BGH$  are also equilateral triangles. Hence  $HG = GB = c$ . Moreover,  $DG = b$ . Therefore  $HD = b + c$ . But  $HD = AE$  since  $FH = FA$  and  $FD = FE$ . Also  $AH = a - BH = a - BG = a - c$ . Hence  $ED = EF = EA + AF = b + c + AH = (b + c) + (a - c) = b + a$ .

We have five possibilities:

- (1)  $b < c < a < b + c < a + b$ ;

- (2)  $c < b < a < b + c < a + b$ ;
- (3)  $c < a < b < b + c < a + b$ ;
- (4)  $b < c < b + c < a < a + b$ ;
- (5)  $c < b < b + c < a < a + b$ .

In (1), we see that  $c < a < b + c$  are three consecutive integers provided  $b = 2$ . Hence we get  $c = 3$  and  $a = 4$ . In this case  $b + c = 5$  and  $a + b = 6$  so that we have five consecutive integers 2, 3, 4, 5, 6 as side lengths. In (2),  $b < a < b + c$  form three consecutive integers only when  $c = 2$ . Hence  $b = 3$ ,  $a = 4$ . But then  $b + c = 5$  and  $a + b = 7$ . Thus the side lengths are 2, 3, 4, 6, 7 which are not consecutive integers. In case (3),  $b < b + c$  are two consecutive integers so that  $c = 1$ . Hence  $a = 2$  and  $b = 3$ . We get  $b + c = 4$  and  $a + b = 5$  so that the consecutive integers 1, 2, 3, 4, 5 form the side lengths. In case (4), we have  $c < b + c$  as two consecutive integers and hence  $b = 1$ . Therefore  $c = 2$ ,  $b + c = 3$ ,  $a = 4$  and  $a + b = 5$  which is admissible. Finally, in case (5) we have  $b < b + c$  as two consecutive integers, so that  $c = 1$ . Thus  $b = 2$ ,  $b + c = 3$ ,  $a = 4$  and  $a + b = 6$ . We do not get consecutive integers.

Therefore the only possibilities are  $(a, b, c) = (4, 2, 3)$ ,  $(2, 3, 1)$  and  $(4, 1, 2)$ . This shows that  $a + b + c = 9, 6$  or  $7$ . Thus there are three possible sums  $AB + BC + CA$ , namely, 6, 7 or 9.

**Solution 2:** As in the earlier solution,  $ED = d = a + b$  and  $EA = e = b + c$ . Let the sides be  $x - 2, x - 1, x, x + 1, x + 2$ . Then  $x \geq 3$ . We also have  $x + 2 \geq x - 1 + x - 2$  so that  $x \leq 5$ . Thus  $x = 3, 4$  or  $5$ . If  $x = 5$ , the sides are  $\{3, 4, 5, 6, 7\}$  and here we do not have two pairs which add to a number in the set. Hence  $x = 3$  or  $4$  and we get the sets as  $\{1, 2, 3, 4, 5\}$  or  $\{2, 3, 4, 5, 6\}$ . With the set  $\{1, 2, 3, 4, 5\}$  we get

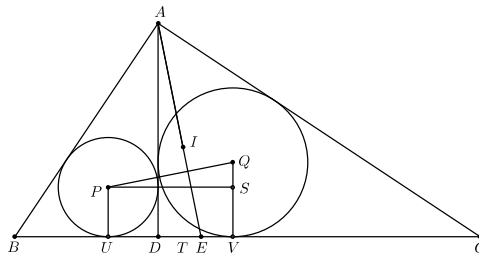
$$(a, b, c, d, e) = (2, 3, 1, 5, 4), (4, 1, 2, 5, 3).$$

From the set  $\{2, 3, 4, 5, 6\}$ , we get  $(a, b, c, d, e) = (4, 2, 3, 6, 5)$ . Thus we see that  $a + b + c = 6, 7$  or  $9$ .

**Solution 3:** We use the same notations and we get  $d = a + b$  and  $e = b + c$ . If  $a \geq 5$ , we see that  $d - b \geq 5$ . But the maximum difference in a set of 5 consecutive integers is 4. Hence  $a \leq 4$ . Similarly, we see  $b \leq 4$  and  $c \leq 4$ . Thus we see that  $a + b + c \leq 2 + 3 + 4 = 9$ . But  $a + b + c \geq 1 + 2 + 3 = 6$ . It follows that  $a + b + c = 6, 7, 8$  or  $9$ . If we take  $(a, b, c, d, e) = (1, 3, 2, 4, 5)$ , we get  $a + b + c = 6$ . Similarly,  $(a, b, c, d, e) = (2, 1, 4, 3, 5)$  gives  $a + b + c = 7$ . For  $a + b + c = 8$ , the only we we can get  $1 + 3 + 4 = 8$ . Here we cannot accommodate 2 and consecutiveness is lost. For 9, we can have  $(a, b, c, d, e) = (3, 2, 4, 5, 6)$  and  $a + b + c = 9$ .

5. Let  $ABC$  be a triangle with  $\angle A = 90^\circ$  and  $AB < AC$ . Let  $AD$  be the altitude from  $A$  on to  $BC$ . Let  $P, Q$  and  $I$  denote respectively the incentres of triangles  $ABD$ ,  $ACD$  and  $ABC$ . Prove that  $AI$  is perpendicular to  $PQ$  and  $AI = PQ$ .

**Solution:** Draw  $PS \parallel BC$  and  $QS \parallel AD$ . Then  $PSQ$  is a right-angled triangle with  $\angle PSQ = 90^\circ$ . Observe that  $PS = r_1 + r_2$  and  $SQ = r_2 - r_1$ , where  $r_1$  and  $r_2$  are the inradii of triangles  $ABD$  and  $ACD$ , respectively. We observe that triangles  $DAB$  and  $DCA$  are similar to triangle  $ACB$ .



Hence

$$r_1 = \frac{c}{a}r, \quad r_2 = \frac{b}{a}r,$$

where  $r$  is the inradius of triangle  $ABC$ . Thus we get

$$\frac{PS}{SQ} = \frac{r_2 + r_1}{r_2 - r_1} = \frac{b + c}{b - c}.$$

On the otherhand  $AD = h = bc/a$ . We also have  $BE = ca/(b + c)$  and

$$BD^2 = c^2 - h^2 = c^2 - \frac{b^2c^2}{a^2} = \frac{c^4}{a^2}.$$

Hence  $BD = c^2/a$ . Therefore

$$DE = BE - BD = \frac{ca}{b + c} - \frac{c^2}{a} = \frac{cb(b - c)}{a(b + c)}.$$

Thus we get

$$\frac{AD}{DE} = \frac{b + c}{b - c} = \frac{PS}{SQ}.$$

Since  $\angle ADE = 90^\circ = \angle PSQ$ , we conclude that  $\triangle ADE \sim \triangle PSQ$ . Since  $AD \perp PS$ , it follows that  $AE \perp PQ$ .

We also observe that

$$PQ^2 = PS^2 + SQ^2 = (r_2 + r_1)^2 + (r_2 - r_1)^2 = 2(r_1^2 + r_2^2).$$

However

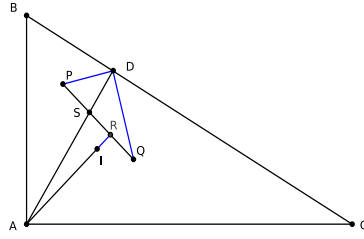
$$r_1^2 + r_2^2 = \frac{c^2 + b^2}{a^2}r^2 = r^2.$$

Hence  $PQ = \sqrt{2}r$ . We also observe that  $AI = r \operatorname{cosec}(A/2) = r \operatorname{cosec}(45^\circ) = \sqrt{2}r$ . Thus  $PQ = AI$ .

**Solution 2:** In the figure, we have made the construction as mentioned in the hint. Since  $P, Q$  are the incentres of  $\triangle ABD, \triangle ACD$ ,  $DP, DQ$  are the internal angle bisectors of  $\angle ADB, \angle ADC$  respectively. Since  $AD$  is the altitude on the hypotenuse  $BC$  in  $\triangle ABC$ , we have that  $\angle PDQ = 45^\circ + 45^\circ = 90^\circ$ . It also implies that

$$\triangle ABC \sim \triangle DBA \sim \triangle DAC$$

This implies that all corresponding length in the above mentioned triangles have the same ratio.



In particular,

$$\begin{aligned}
 \frac{AI}{BC} &= \frac{DP}{AB} = \frac{DQ}{AC} \\
 \Rightarrow \frac{AI^2}{BC^2} &= \frac{DP^2}{AB^2} = \frac{DQ^2}{AC^2} = \frac{DP^2 + DQ^2}{AB^2 + AC^2} \\
 \Rightarrow \frac{AI^2}{BC^2} &= \frac{PQ^2}{BC^2}, \quad \text{by Pythagoras Theorem in } \triangle ABC, \triangle PDQ \\
 \Rightarrow AI &= PQ
 \end{aligned}$$

as required.

For the second, part, we note that from the above relations, we have  $\triangle ABC \sim \triangle DPQ$ . Let us take  $\angle ACB = \theta$ . Then, we get

$$\begin{aligned}
 \angle PSD &= 180^\circ - (\angle SPD + \angle SDP) \\
 &= 180^\circ - (90^\circ - \theta + 45^\circ) \\
 &= 45^\circ + \theta
 \end{aligned}$$

This gives us that

$$\begin{aligned}
 \angle ARS &= 180^\circ - (\angle ASR + \angle SAR) \\
 &= 180^\circ - (\angle PSD + \angle SAC - \angle IAC) \\
 &= 180^\circ - (45^\circ + \theta + 90^\circ - \theta - 45^\circ) \\
 &= 90^\circ
 \end{aligned}$$

as required. Hence, we get that  $AI = PQ$  and  $AI \perp PQ$ .

**Solution 3:** We know that the angle bisector of  $\angle B$  passes through  $P, I$  which implies that  $B, P, I$  are collinear. Similarly,  $C, Q, I$  are also collinear. Since  $I$  is the incentre of  $\triangle ABC$ , we know that

$$\angle PIQ = \angle BIC = 90^\circ + \frac{\angle A}{2} = 135^\circ$$

Join  $AP, AQ$ . We know that  $\angle BAP = \frac{1}{2}\angle BAD = \frac{1}{2}\angle C$ . Also,  $\angle ABP = \frac{1}{2}\angle B$ . Hence by Exterior Angle Theorem in  $\triangle ABP$ , we get that

$$\angle API = \angle ABP + \angle BAP = \frac{1}{2}(\angle B + \angle C) = 45^\circ$$

Similarly in  $\triangle ADC$ , we get that  $\angle AQI = 45^\circ$ . Also, we have

$$\angle PAI = \angle BAI - \angle BAP = 45^\circ - \frac{\angle C}{2} = \frac{\angle B}{2}$$

Similarly, we get  $\angle QAI = \frac{\angle C}{2}$ .

Now applying Sine Rule in  $\triangle API$ , we get

$$\frac{IP}{\sin \angle PAI} = \frac{AI}{\sin \angle API} \implies IP = \sqrt{2}AI \sin \frac{B}{2}$$

Similarly, applying Sine Rule in  $\triangle AQI$ , we get

$$\frac{IQ}{\sin \angle PAI} = \frac{AI}{\sin \angle AQI} \implies IQ = \sqrt{2}AI \sin \frac{C}{2}$$

Applying Cosine Rule in  $\triangle PIQ$  gives us that

$$\begin{aligned} PQ^2 &= IP^2 + IQ^2 - 2 \cdot IP \cdot IQ \cos \angle PIQ \\ &= 2AI^2 \left( \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + \sqrt{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \end{aligned}$$

We will prove that  $(\sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + \sqrt{2} \sin \frac{B}{2} \sin \frac{C}{2}) = \frac{1}{2}$ . In any  $\triangle XYZ$ , we have that

$$\sum_{cyc} \sin^2 \frac{X}{2} = 1 - 2 \prod \sin \frac{X}{2}$$

Using this in  $\triangle ABC$ , and using the fact that  $\angle A = 90^\circ$ , we get

$$\begin{aligned} \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} &= 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ \implies \frac{1}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} &= 1 - \sqrt{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ \implies \left( \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + \sqrt{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) &= \frac{1}{2} \end{aligned}$$

which was to be proved. Hence we get  $PQ = AI$ .

The second part of the problem can be obtained by angle-chasing as outlined in Solution 2.

**Solution 4:** Observe that  $\angle APB = \angle AQC = 135^\circ$ . Thus  $\angle API = \angle AQI = 45^\circ$  (since  $B - P - I$  and  $C - Q - I$ ). Note  $\angle PAQ = 1/2\angle A = 45^\circ$ . Let  $X = BI \cap AQ$  and  $Y = CI \cap AP$ . Therefore  $\angle AXP = 180 - \angle API - \angle PAQ = 90^\circ$ . Similarly  $\angle AYQ = 90^\circ$ . Hence  $I$  is the orthocentre of triangle  $PAQ$ . Therefore  $AI$  is perpendicular to  $PQ$ . Also  $AI = 2R_{PAQ} \cos 45^\circ = 2R_{PAQ} \sin 45^\circ = PQ$ .

6. Let  $n \geq 1$  be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \dots$$

Show that  $2x - 1, 2x, 2x + 1$  form the sides of a triangle whose area and inradius are also integers.

**Solution:** Consider the binomial expansion of  $(2 + \sqrt{3})^n$ . It is easy to check that

$$(2 + \sqrt{3})^n = x + y\sqrt{3},$$

where  $y$  is also an integer. We also have

$$(2 - \sqrt{3})^n = x - y\sqrt{3}.$$

Multiplying these two relations, we obtain  $x^2 - 3y^2 = 1$ .

Since all the terms of the expansion of  $(2 + \sqrt{3})^n$  are positive, we see that

$$2x = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n = 2 \left( 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \dots \right) \geq 4.$$

Thus  $x \geq 2$ . Hence  $2x + 1 < 2x + (2x - 1)$  and therefore  $2x - 1, 2x, 2x + 1$  are the sides of a triangle. By Heron's formula we have

$$\Delta^2 = 3x(x+1)(x-1) = 3x^2(x^2 - 1) = 9x^2y^2.$$

Hence  $\Delta = 3xy$  which is an integer. Finally, its inradius is

$$\frac{\text{area}}{\text{perimeter}} = \frac{3xy}{3x} = y,$$

which is also an integer.

**Solution 2:** We will first show that the numbers  $2x_n - 1, 2x_n, 2x_n + 1$  form the sides of a triangle. To show that, it suffices to prove that  $2x_n - 1 + 2x_n > 2x_n + 1$ . If possible, let the converse hold. Then, we see that we must have  $4x_n - 1 \leq 2x_n + 1$ , which implies that  $x_n \leq 1$ . But we see that even for the smallest value of  $n = 1$ , we have that  $x_n > 1$ . Hence, the numbers are indeed sides of a triangle.

Let  $\Delta_n, r_n, s_n$  denote respectively, the area, inradius and semiperimeter of the triangle with sides  $2x_n - 1, 2x_n, 2x_n + 1$ . By Heron's Formula for the area of a triangle, we see that

$$\Delta_n = \sqrt{3x_n(x_n - 1)x_n(x_n + 1)} = x_n \sqrt{3(x_n^2 - 1)}$$

If possible, let  $\Delta_n$  be an integer for all  $n \in \mathbb{N}$ . We see that due to the presence of the first term  $\binom{n}{0} 2^n$ , we have  $3 \nmid x_n, \forall n \in \mathbb{N}$ . Hence, we get that  $3 \nmid x_n^2 - 1$ . Hence, we can write  $x_n^2 - 1$  as  $3m$  for some  $m \in \mathbb{N}$ . Then, we can also write

$$\Delta_n = 3x_n \sqrt{m}$$

Note that we have assumed that  $\Delta_n$  is an integer. Hence, we see that we must have  $m$  to be a perfect square. Consequently, we get that

$$r_n = \frac{\Delta_n}{s_n} = \frac{\Delta_n}{3x_n} = \sqrt{m} \in \mathbb{Z}$$

Hence, it only remains to show that  $\Delta_n \in \mathbb{Z}$ ,  $\forall n \in \mathbb{N}$ . In other words, it suffices to show that  $3(x_n^2 - 1)$  is a perfect square for all  $n \in \mathbb{N}$ .

We see that we can write  $x_n$  as

$$\begin{aligned}
 x_n &= \frac{1}{2} \left( 2 \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k \right) \\
 &= \frac{1}{2} \left( (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \right) \\
 3x_n^2 - 3 &= \frac{3}{4} \left( (2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n} + 2(2 + \sqrt{3})^n (2 - \sqrt{3})^n \right) - 3 \\
 &= \frac{3}{4} \left( (2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n} - 2(2 + \sqrt{3})^n (2 - \sqrt{3})^n \right) \\
 &= \left( \frac{\sqrt{3}}{2} \left( (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right) \right)^2
 \end{aligned}$$

We are left to show that the quantity obtained in the above equation is an integer. But we see that if we define

$$a_n = \frac{\sqrt{3}}{2} \left( (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right), \quad \forall n \in \mathbb{N}$$

the sequence  $\langle a_k \rangle_{k=1}^{\infty}$  thus obtained is exactly the solution for the recursion given by

$$a_{n+2} = 4a_{n+1} - a_n, \quad \forall n \in \mathbb{N}, \quad a_1 = 3, a_2 = 12$$

Hence, clearly, each  $a_n$  is obviously an integer, thus completing the proof.

————-000000————-